FALL 2024

MATH 210: Multivariable Calculus

Chapter 5: Vector Calculus

Line Integrals of Scalar Functions:

- given $f(x,y)$ and $\vec{r}'(t)$.

$$\text{aggregation} \leftarrow A = \int_C \overset{\swarrow \text{phenomenon}}{f(x,y)}\ ds \leftarrow \text{path}$$

$$= \int_{t=a}^{b} f(\vec{r}(t))\ \underbrace{|\vec{r}'(t)|}_{\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}}\ dt$$

↓ parametrize

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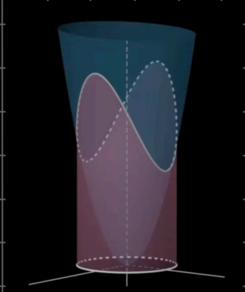
1 Evaluate $\int_C (2x^2+y^2)\ ds$ counterclockwise over the circle of radius 2 centered at the origin.

$$A = \int_C f(\vec{r}(t))\ |r'(t)|\ dt$$



$$f(x,y) = 2x^2+y^2$$
$$\vec{r}(t) = \langle 2\cos t, 2\sin t\rangle \quad 0 \le t \le 2\pi$$
$$\vec{r}'(t) = \langle -2\sin t, 2\cos t\rangle$$
$$|\vec{r}'(t)| = 2$$
$$f(\vec{r}(t)) = 2(2\cos t)^2 + (2\sin t)^2$$
$$= 8\cos^2 t + 4\sin^2 t$$
$$= 4(2\cos^2 t + \sin^2 t) = 4(\cos^2 t + \cos^2 t + \sin^2 t)$$
$$= 4(\cos^2 t + 1) = 4\left(\frac{1+\cos(2t)}{2} + 1\right)$$
$$= 4\left(\frac{\cos(2t)+3}{2}\right) = 2(\cos(2t)+3)$$

$$\therefore A = 2\int \cos(2t) + 3 \cdot 2dt$$
$$= 4 \cdot \frac{\sin(2t)}{2} + 3t \Big]_0^{2\pi}$$
$$= 4 \cdot 6\pi = 24\pi \ //$$

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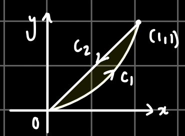
2. $\int_C 2x\ ds$ along $y = x^2$ from $(0,0) \to (1,1) \to (0,0)$ along $y = x$

$$A = \int_a^b f(\vec{r}(t))\ \|\vec{r}'(t)\|$$
$$C_1 : \vec{r}(t) = \langle t, t^2\rangle \quad 0 \le t \le 1$$
$$C_2 : \vec{r}(t) = \langle 1-t, 1-t\rangle \quad 0 \le t \le 1$$

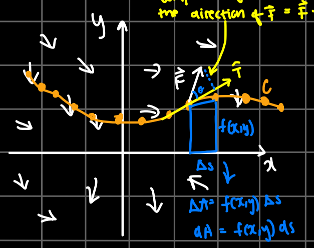


$$A = \int_{C_1} 2x\ ds + \int_{C_2} 2x\ ds$$
$$= \int_0^1 2t\sqrt{1+4t^2}\ dt + \int_0^1 2 - 2t(\sqrt{2})\ dt$$
$$= \frac{1}{4}\int_0^1 u^{1/2}\ du + \int_0^1 2\sqrt{2} - 2\sqrt{2}t\ dt$$
$$= \frac{2}{12}\left[(1+4t^2)^{3/2}\right]_0^1 + 2\sqrt{2}t - \sqrt{2}t^2 \Big]_0^1$$
$$= \frac{1}{6}(5\sqrt{5}-1) + \sqrt{2} \ //$$

$u = 1+4t^2$
$du = 8t\ dt$
$2t\ dt = \frac{du}{4}$

Line Integrals through Vector Fields

component of F in the direction of $\vec{T} = \vec{F}\cdot\hat{T} = |\vec{F}||\hat{T}|\cos\theta = |\vec{F}|\cos\theta$



- Let $\vec{F}$ be a continuous vector field $\vec{F}$ defined on a smooth curve C, given by $\vec{r}(t)$ for $a \le t \le b$.
- Then, the line integral of $\vec{F}$ along C is:

$$- \int_C \vec{F}\cdot dr = \int_C \vec{F}\cdot\hat{T}\ ds = \int_C \vec{F}\cdot r'(t)\ dt = \int_C M dx + N dy$$

$\hat{T} = \frac{r'(t)}{\cancel{|r'(t)|}}$, $ds = \cancel{|r'(t)|}\ dt$

$r'(t) = \langle \frac{dx}{dt}, \frac{dy}{dt}\rangle$

$\vec{F}\cdot\hat{T}$: how much work is being done at each point

$\vec{F} = \langle M, N\rangle$

- The line integral measures the work done by a vector field along a path.

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1. $\vec{F}(x,y) = xy^2\hat{i} - x^2\hat{j} = \langle xy^2, -x^2\rangle$

$$\vec{r}(t) = t^3\hat{i} + t^2\hat{j} = \langle t^3, t^2\rangle \quad 0 \le t \le 1$$
$$\vec{r}'(t) = \langle 3t^2, 2t\rangle$$

$$\left.\begin{aligned}
&\text{Let } x = t^3,\ y = t^2 \\
&\therefore \vec{F} = \langle t^3(t^2)^2, -(t^3)^2\rangle \\
&\quad\ = \langle t^7, -t^6\rangle \\
&\vec{F}\cdot\vec{r}'(t) = 3t^9 - 2t^7
\end{aligned}\right\}\ \text{parametrize}$$

Line Integral. $\int_0^1 3t^9 - 2t^7\ dt = \frac{3}{10}t^{10} - \frac{1}{4}t^8\Big]_0^1$
$$= \frac{3}{10} - \frac{1}{4}$$
$$= \frac{2}{40} = \frac{1}{20} \ //$$

↓ slight boost on $\vec{r}(t)$ from $\vec{F}$

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2. $\vec{F}(x,y,z) = \langle \sin(x), \cos(y), xz\rangle$

$$\vec{r}(t) = \langle t^2, t^3, -2t\rangle, \quad 0 \le t \le 2$$
$$\vec{r}'(t) = \langle 2t, 3t^2, -2\rangle$$
$$\vec{F}(t) = \langle \sin(t^2), \cos(t^3), -2t^3\rangle$$
$$\vec{F}(t)\cdot\vec{r}'(t) = 2t\cdot\sin(t^2) + 3t^2\cos(t^3) + 4t^3$$

$$L = \int_0^2 2t\sin(t^2) + \int_0^2 3t^2\cos(t^3) + \int_0^2 4t^3\ dt$$
$$= -\cos(t^2) + \sin(t^3) + t^4\Big]_0^2$$
$$= -\cos(4) + \sin(8) + 16 + \cos(0) - \cancel{\sin(0)}^{\,0}$$
$$= -\cos(4) + \sin(8) + 17 \ //$$

OR

$$\int_C \vec{F}\cdot d\vec{r} = \int_C F_1\,dx + F_2\,dy + F_3\,dz$$

3. $\vec{F} = \langle y, x \rangle$ C is $y = 2x^2$ from $(0,0)$ to $(2,8)$

$$W = \int_C \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$\vec{r}(t) = \langle t, 2t^2 \rangle$ $0 \le t \le 2$

$\vec{r}\,'(t) = \langle 1, 4t \rangle$

$\vec{F}(\vec{r}(t)) = \langle 2t^2, t \rangle$

$\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = 2t^2 + 4t^2 = 6t^2$

$$\therefore W = \int_0^2 6t^2\, dt = 2t^3\Big]_0^2 = 16 \,//$$

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4. $\vec{F} = \langle yz, xz, xy \rangle$

$\vec{r}(t) = \langle t^2, t, t^3 \rangle$ $0 \le t \le 2$

$$W = \int_C F(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$\vec{r}\,'(t) = \langle 2t, 1, 3t^2 \rangle$

$\vec{F}(\vec{r}(t)) = \langle t^4, t^5, t^3 \rangle$

$\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = 2t^5 + t^5 + 3t^5 = 6t^5$

$$\therefore W = \int_0^2 6t^5 = t^6\Big]_0^2 = 64 \,//$$

Circulation

- If a line integral of a vector field $\vec{F}$ is along an oriented, closed curve, then

  how much they work together

  $\vec{F} = \langle M, N \rangle$

  $$\text{closed curve} \rightarrow \oint_C \vec{F} \cdot \hat{T}\, ds = \oint_C M\,dx + N\,dy$$

  is called the flow/circulation of $\vec{F}$, and it measures the tendency of something to flow in the direction of curve C.

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1 Calculate the circulation of $\vec{F} = \langle -y, x \rangle$ along the circle of radius 2 centered at the origin oriented counter-clockwise.

$$C = \oint_C \vec{F} \cdot \hat{T}\, ds = \oint_C \vec{F} \cdot \vec{r}\,'(t)\, dt$$

$\vec{r}(t) = \langle \overset{M}{2\cos t}, \overset{N}{2\sin t} \rangle$ $0 \le t \le 2\pi$

$\vec{r}\,'(t) = \langle -2\sin t, 2\cos t \rangle$

$\vec{F}(\vec{r}(t)) = \langle -2\sin t, 2\cos t \rangle$

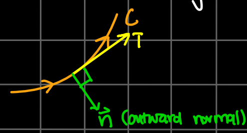
$\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = 4\sin^2 t + 4\cos^2 t = 4$

$$\therefore C = \int_0^{2\pi} 4\, dt = 4t\Big]_0^{2\pi} = 8\pi \,//$$

Flux

clockwise rotation of the tangent

- If a path is some $\vec{r} = \langle x, y \rangle$, then $\vec{n} = \langle y', -x' \rangle$ is normal to $\vec{r}$ and points to the right as we travel along the curve.



to what extent are we crossing across the field?

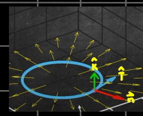
- Flux is the measure of flow through a curve so the formula is

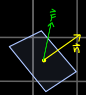
$$\vec{n} = \vec{T} \times \hat{k} = \begin{vmatrix} \frac{dx}{ds} & \frac{dy}{ds} & 0 \\ 0 & 0 & 1 \end{vmatrix} = \frac{dy}{ds} - \frac{dx}{ds} = \langle y', -x' \rangle$$

$$\oint_C \vec{F} \cdot \hat{N}\, ds = \oint_C \vec{F}\, \frac{\vec{n}(t)}{\cancel{|\vec{n}(t)|}} \cdot \cancel{|\vec{r}\,'(t)|}\, dt$$

$|\vec{r}\,'(t)|\, dt$ $\frac{\vec{n}(t)}{|\vec{n}(t)|}$ equivalent

proportion of the field in the normal direction





$$\int_C \vec{F} \cdot \vec{n}(t)\, dt = \int_C M\,dy - N\,dx$$

$\langle M, N \rangle$ $\langle y', -x' \rangle$

$\langle \frac{dy}{dt}, -\frac{dx}{dt} \rangle$

OR

$$\int_C \vec{F} \times \vec{T}\, ds = \int_C \vec{F} \times \vec{r}\,'(t)\, dt$$

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1. Calculate the flux of $\vec{F} = \langle x, y \rangle$ across the circle of radius 3 centered at the origin oriented counterclockwise.

$$\int_C \vec{F} \cdot \vec{n}\, dt = \int M\,dy - N\,dx$$

$\vec{r}(t) = \langle 3\cos t, 3\sin t \rangle$ $0 \le t \le 2\pi$

$\vec{F}(\vec{r}(t)) = \langle 3\cos t, 3\sin t \rangle$

$\vec{r}\,'(t) = \langle -3\sin t, 3\cos t \rangle$

$\vec{F} \times \vec{r}\,' = 9\cos^2 t + 9\sin^2 t = 9$

$$\therefore \int_0^{2\pi} 9\, dt = 9\Big]_0^{2\pi} = 18\pi \,//$$

Fundamental Theorem of Line Integrals:

- Recall FTOC: $\int_a^b f'(x)\,dx = f(b) - f(a)$

- (Theorem): $\int_a^b \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$

- (Definition): Let $\vec{F}$ be a vector field.
  - We say $\vec{F}$ is conservative if $\exists f(x,y)$ such that $\vec{F} = \nabla f$
    - a gradient field
    - path independent

- Given $\vec{F}$, how can we determine if $\vec{F}$ is conservative?

1. $\vec{F} = \langle -x, y \rangle$. Conservative?

$f_x = -x \Rightarrow -\frac{1}{2}x^2 + c(y)$

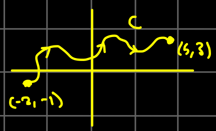
$f_y = y \Rightarrow \frac{1}{2}y^2 + c(x)$

$f(x,y) = -\frac{1}{2}x^2 + \frac{1}{2}y^2$

check: $\nabla f = \langle -x, y \rangle$

$\therefore \vec{F}$ is conservative.

Ex: Compute the line integral of C over $\vec{F} = \langle -x, y \rangle$



$$\int_C \vec{F} \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$
$$= f(5,3) - f(-3,-1)$$
$$= \left(-\frac{25}{2} + \frac{9}{2}\right) - \left(-\frac{9}{2} + \frac{1}{2}\right)$$
$$= -13 + 8 = -5$$

OR → in $\mathbb{R}^3$, if curl $\vec{F}$ $(\nabla \times \vec{F}) = 0$

- We can use Clairaut's Theorem: $f_{xy} = f_{yx}$.
- Ex:

$\vec{F} = \langle -x, y \rangle$ (M, N)

$\frac{\partial}{\partial y} M = f_{xy} = 0$

$\frac{\partial}{\partial x} N = f_{yx} = 0$

$f_{xy} = f_{yx} = 0 \quad \therefore \vec{F}$ is conservative.

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2. $\vec{F} = \langle y, x \rangle$

$\frac{\partial}{\partial y} F_1 = 1$

$\frac{\partial}{\partial x} F_2 = 1$

$\therefore \vec{F}$ is conservative.

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3. $\vec{F} = \langle 2xy + y^2, x^2 + 2xy \rangle$

$\frac{\partial}{\partial y} F_1 = 2x + 2y$

$\frac{\partial}{\partial x} F_2 = 2x + 2y$

$\therefore \vec{F}$ is not conservative.

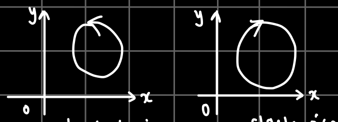
(Corollary) Let C be a loop and let $\vec{F}$ be conservative.

Then, $\oint_C \vec{F} \cdot d\vec{r} = 0$

$\int_a^b \vec{F} \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$

But since this is a loop, $a = b$.

Green's Theorem:

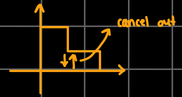


counterclockwise positive | clockwise negative

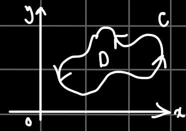
\# C is a positive-oriented loop

if not: negate answer.

\# Loops can be manufactured.



- A non-conservative field will not "balance" with a loop.



- If $\vec{F}$ has a positive effect, then there has to be circulation around my loop?

- Green's Theorem relies on measuring the circulation (or curl) inside the loop to measure the effect of $\vec{F}$ on C

Circulation / Curl Form:

simple, closed curve → $\oint_C \vec{F} \cdot \hat{T}\,ds = \iint_R \underbrace{(N_x - M_y)}_{\text{2-D curl / circulation density}}\,dA$ (dx dy / r dr dθ)

Flux Form:

$$\oint \vec{F} \cdot \hat{N}\,ds = \iint_R \underbrace{(M_x + N_y)}_{\text{2-D divergence / flux density}}\,dA$$

where $\vec{F} = \langle M, N \rangle$

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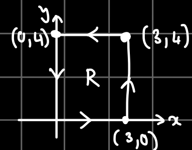
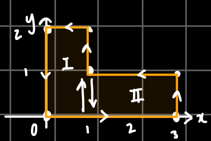
1. $\vec{F} = \langle ye^x, 2e^x \rangle$ (M, N), $C: (0,0) \to (3,0) \to (3,4) \to (0,4) \to (0,0)$



$$\int_C \vec{F} \cdot d\vec{r} = \int_0^4 \int_0^3 (2e^x - e^x)\,dx\,dy$$
$$= \int_0^4 \int_0^3 e^x\,dx\,dy$$
$$= \int_0^4 e^3 - 1\,dy = 4(e^3 - 1)$$

---

2. $\vec{F} = \langle xy, xy^2 \rangle$ (M, N) over $(0,0) \to (3,0) \to (3,1) \to (1,1) \to (1,2) \to (0,2) \to (0,0)$

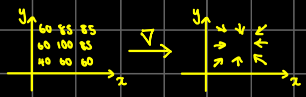


$$\int_C \vec{F} \cdot d\vec{r} = \iint N_x - M_y\,dA$$
$$= \iint_D (y^2 - x)\,dx\,dy$$
$$= \int_0^2 \int_0^1 y^2 - x\,dx\,dy + \int_0^1 \int_1^3 y^2 - x\,dx\,dy$$
$$= \int_0^2 xy^2 - \frac{1}{2}x^2\Big]_0^1 dy + \int_0^1 xy^2 - \frac{1}{2}x^2\Big]_1^3 dy$$
$$= \int_0^2 y^2 - \frac{1}{2}\,dy + \int_0^1 2y^2 - 4\,dy$$
$$= \frac{1}{3}y^3 - \frac{1}{2}y\Big]_0^2 + \frac{2}{3}y^3 - 4y\Big]_0^1$$
$$= \frac{8}{3} - 1 + \frac{2}{3} - 4$$
$$= \frac{10}{3} - 5$$
$$= -\frac{5}{3}$$

Divergence → Dot Product

- Recall, $\text{grad}(f) = \nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$ or $\langle f_x, f_y, f_z \rangle$

scalar function $\xrightarrow{\nabla}$ vector field



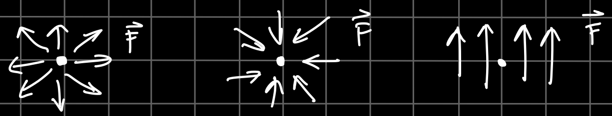
- The divergence of a vector field is

$\text{div}(\vec{F}) = \nabla \cdot \vec{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \langle M, N, P \rangle$

$= M_x + N_y + P_z \longrightarrow$ flux through a surface

2-D flux from Green's Theorem

vector field $\xrightarrow{\nabla \cdot}$ scalar function



net emission $\text{div}\vec{F}(x,y,z) > 0$ | net absorption $\text{div}\vec{F}(x,y,z) < 0$ | neutral $\text{div}\vec{F}(x,y,z) = 0$

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1. $\vec{F} = \langle xy, xy^2, 0 \rangle$ Divergence at $(0,0,0)$?

$\text{div}\,\vec{F} = y + 2xy + 0$

$\text{div}\,\vec{F}_{(0,0,0)} = 0 \quad \therefore$ neutral.

What about at $(1,-1,1)$?

$\text{div}\,\vec{F}_{(1,-1,1)} = -1 - 2 = -3 \quad \therefore$ net absorption

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2. Calculate $\nabla \cdot \vec{F}$ given $\vec{F} = \langle x^2, y \rangle$

$\nabla \cdot \vec{F} = 2x + 1 \rightarrow$ variable

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3. Calculate $\nabla \cdot \vec{F}$ given $\vec{F} = \langle x, 2y, 3z \rangle$

$\nabla \cdot \vec{F} = 1 + 2 + 3 = 6 \rightarrow$ constant net emission

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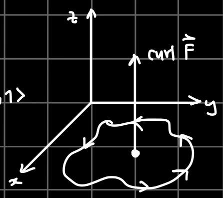
Curl → Cross Product



- The curl of a vector field is

$\text{curl}(\vec{F}) = \nabla \times \vec{F} = (N_x - M_y)\hat{k}$ ← $\langle 0,0,1 \rangle$

2-D curl

- $\nabla \times \vec{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \times \langle M, N, P \rangle$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$$

vector field $\xrightarrow{\nabla \times}$ vector field

- Curl is a measure of how much a vector field is rotating about a point or direction.

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1. Calculate $\nabla \times \vec{F}$ given $\vec{F} = \langle yz, xz, xy \rangle$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix}$$

$= \langle x - x, y - y, z - z \rangle = \langle 0, 0, 0 \rangle = \vec{0}$

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2. Calculate $\nabla \times \vec{F}$ given $\vec{F} = \langle 0, x \rangle$

$\nabla \times \vec{F} = (N_x - M_y)\hat{k}$

$= (1 - 0)\langle 0, 0, 1 \rangle$

$= \langle 0, 0, 1 \rangle = \hat{k}$