SPRING 2026

MATH 214: Ordinary Differential Equations

Monday, January 12

chapter 1: Basic Introductions

chapter 1.1: Basic Mathematical Models; Direction Fields

example: $\frac{dv}{dt} = 9.8 - \frac{v}{5}$

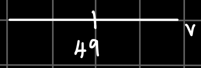
1. Let $\frac{dv}{dt} = 0$ and solve for v.

$\Rightarrow 0 = 9.8 - \frac{v}{5}$

$\Rightarrow v = 9.8 \times 5$

$= 49$ m/s

inflection point



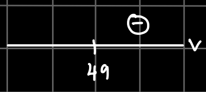
2. check both sides of inflection point.

a. let $v = 50$

$\Rightarrow \frac{dv}{dt} = 9.8 - \frac{50}{5}$

$= 9.8 - 10$

$= -0.2 < 0$

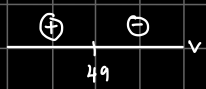


b. let $v = 45$

$\Rightarrow \frac{dv}{dt} = 9.8 - \frac{45}{5}$

$= 9.8 - 9$

$= 0.8 > 0$



Wednesday, January 14

example: $y'(t) = \frac{1}{2}(3 - y)$

1. set $y' = 0$ and solve for y
   - $0 = \frac{1}{2}(3 - y)$

   $\Rightarrow y = 3$

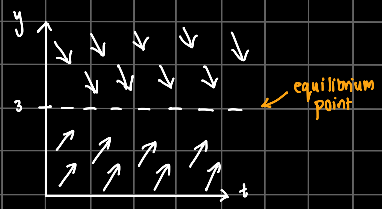
2. Let $y > 3 = 4$

   $\Rightarrow y' = 1/2(3-4) < 0$ ↘↘

3. Let $y < 3 = 2$

   $\Rightarrow y' = 1/2(3-2) > 0$ ↗↗

4. graph



example: $y'(t) = (y-1)(y-5)$

1. Let $y' = 0$:

   $\Rightarrow y = 1$ and $y = 5$

2. Let $y > 5 = 6$.

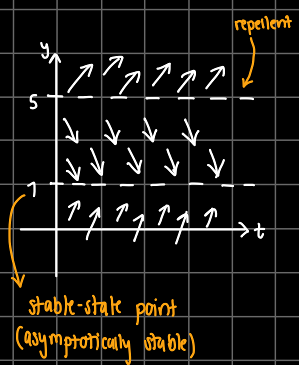
   $\Rightarrow y'(t) > 0$

3. Let $1 < y < 5 = 3$

   $\Rightarrow y'(t) < 0$

4. Let $y < 1 = 0$

   $\Rightarrow y'(t) > 0$



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definition: differential equation

- a differential equation is an equation which contains derivatives of the unknown.
- there are two classes of diff. eqs.
  1. ODE (ordinary diff. eqs)
  2. PDE (partial diff. eqs)
- the order of the diff. eq. is the highest derivative.
  - $y' \rightarrow y'' \rightarrow y''' \rightarrow y^{(n)}$
- a linear diff eq. is written as:

$$\frac{dy}{dx} + Py = Q,$$

where P and Q are functions in x, or

$$\frac{dx}{dy} + Px = Q,$$

where P and Q are functions in y.

example: determine whether the following diff. eqs are linear or non linear.

1. $(y+t)\frac{dy}{dt} + y = 1$

$\Rightarrow \frac{dy}{dt} + \underbrace{\frac{1}{y+t}}_{P}\cdot\boxed{y} = \underbrace{\frac{1}{y+t}}_{Q}$

- since $\frac{1}{y+t}$ is not solely a function of t, the diff. eq is nonlinear.

2. $3y' + (t+4)y = t^2 + y''$

$y'' - 3y' - (t+4)\boxed{y} = -t^2$

- since $(t+4)$ and t^2 are solely functions of t, the diff. eq is linear.

3. $y^{(3)} = \cos(2ty)$

- since $\cos(2ty)$ is not solely a function of t, the diff. eq is nonlinear.

4. $y^{(4)} + \sqrt{t}\,y^{(2)} + \cos(t) = e^y$

- since e^y is not a function of t, the diff. eq is nonlinear

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recall: linear diff. eq

- $\frac{dy}{dx} + Py = Q$

  $y' + Py = Q$ where P, Q are functions of x
- $y'' + Ry' + Py = Q$ where P, Q, R are functions of x.
- $y''' + Zy'' + Ry' + Py = Q$ where P, Q, R, Z are functions of x
- In general,

$$y^{(n)} + a_1(x)y^{(n-1)} + \dots + a_n(x)y = Q$$

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topic: introduction to initial value problems

find derivatives, plug in

example: verify that $y(x) = ce^{2x}$, $y(0) = 3$ is a solution to $y' = 2y$ (1)

- $y'(x) = 2ce^{2x}$ (2)
- plug (2) into (1):

$\Rightarrow y' = 2y$

$= 2(ce^{2x})$

$= y'(x)$

- solve for c:

$\Rightarrow y(0) = ce^{2(0)}$

$3 = c(1)$

$\Rightarrow c = 3$

$\Rightarrow y(x) = 3e^{2x}$

example: solve $y' = 2y$, $y(0) = 3$

- $y' = \frac{dy}{dx} = 2y$

$\Rightarrow \frac{dy}{y} = 2dx$

$\Rightarrow \int \frac{dy}{y} = 2\int dx$

$\Rightarrow \ln|y| = 2x + c$

$\Rightarrow y = e^{2x+c}$

$= e^{2x}e^{c}$ let $c = e^c$

$= ce^{2x}$ ⊠

- $y(x) = ce^{2x}$

$y(0) = ce^{2(0)}$

$\Rightarrow c = 3$

$\Rightarrow y(x) = 3e^{2x}$ ⊠

example: bunnies at $t = 0$, population = 1000 bunnies

rate of population $\propto$ population.

- $P(0) = 1000$
- $\frac{dP}{dt} = kP$

$\frac{dP}{P} = k\,dt$

$\int \frac{dP}{P} = k \int dt$

$\ln |P| = kt + c$

$\Rightarrow P = e^{kt+c}$

$\Rightarrow P(t) = c\,e^{kt}$

- $P(0) = c = 1000$

$\Rightarrow P(t) = 1000\,e^{kt}$ ⊠

example: solve $\frac{dy}{dx} = (x-2)^2$, $y(0) = 3$

- $dy = (x-2)^2\,dx$

$\int dy = \int (x-2)^2\,dx$

- Let $u = x-2 \Rightarrow du = dx$

$\Rightarrow y(x) = \int u^2\,du$

$= \frac{1}{3} u^3 + c$

$y(x) = \frac{1}{3}(x-2)^3 + c$

- $y(0) = \frac{1}{3}(-2)^3 + c$

$3 = \frac{-8}{3} + c$

$\Rightarrow c = \frac{17}{3}$

$\Rightarrow y(x) = \frac{1}{3}(x-2)^3 + \frac{17}{3}$ ⊠

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chapter 2: first order diff. eqs

chapter 2.1: linear eqs: method of integrating factors

recall: first order linear diff. eq.

need coefficient of 1

- $\frac{dy}{dx} + P(x) = Q(x)$

$y' + P(x) = Q(x)$ (1)

- multiply (1) by $\mu(x)$

$(uv)' = u'v + uv'$

$\mu \cdot y' + \underline{\mu(x) P(x)} y = \mu(x) Q(x)$ (2)

$\Rightarrow \frac{d}{dx}(\mu y) = \mu y' + \mu' y$ (3)

- compare (3) with LHS of (2)

$\Rightarrow \mu' = \mu(x) P(x)$

$d\mu/dx = \mu(x) P(x)$

$\Rightarrow d\mu/\mu(x) = P(x)\,dx$

$\int d\mu/\mu(x) = \int P(x)\,dx$

$\ln|\mu(x)| = \int P(x)\,dx$

$\Rightarrow \mu(x) = e^{\int P(x)\,dx}$ (4)

- rewrite (2) with (4)

$\Rightarrow \mu y' + \mu' y = \mu(x) Q(x)$

$(\mu y)' = \mu(x) Q(x)$

$\int (\mu y)' = \int \mu(x) Q(x)\,dx$

$\mu y = \int \mu(x) Q(x)\,dx$

$y(x) = 1/\mu \int \mu(x) Q(x)\,dx$ ⊠

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example: solve $y' + y = e^{2t}$ (1)

- $dy/dt + P(t) y = Q(t)$

where $P(t) = 1$ and $Q(t) = e^{2t}$

$\Rightarrow$ we have a first order linear diff. eq.

- let's use the method of integrating factors.
- $\mu(t) = e^{\int P(t)\,dt}$

$= e^{\int 1\,dt}$

$= e^t$ (2)

- multiply (1) by (2)

$\Rightarrow \mu(t) y' + \mu(t) y = \mu(t) e^{2t}$

$e^t y' + e^t y = e^t e^{2t}$

$(e^t y)' = e^{3t}$ [product rule]

$e^t y = \int e^{3t}\,dt$ [integrate]

$e^t y = \frac{1}{3} e^{3t} + c$

$\Rightarrow y(t) = \frac{1}{3} e^{2t} + c e^{-t}$ ⊠

- verification: plug everything into

$y(x) = 1/\mu \int Q(x) \mu(x)\,dx$

example: solve $ty' - y = t^2 e^{-t}$, $t > 0$

- $ty' - y = t^2 e^{-t}$

$y' - 1/t\, y = t e^{-t}$ [divide by t]

- $\mu(t) = e^{\int P(t)\,dt} = e^{\int -1/t\,dt}$

$= e^{-\ln|t|} = \cancel{e^{\ln}} t^{-1}$

$= t^{-1}$

$\Rightarrow t^{-1}(y') - t^{-1}(1/t\, y) = t^{-1}(t e^{-t})$

$t^{-1} y' - t^{-2} y = e^{-t}$

$(t^{-1} y)' = e^{-t}$

$t^{-1} y = \int e^{-t}\,dt$

$\Rightarrow y(t) = t(-e^{-t} + c)$ ⊠

example: $y' - 1/3\, y = e^{-t}$, $y(0) = a$

- $P(x) = -1/3$, $Q(x) = e^{-t}$
- $\mu(t) = e^{\int -1/3\,dt}$

$= e^{-1/3\, t}$

- $e^{-1/3 t} y' - 1/3\, e^{-1/3 y} y = e^{-4/3 t}$

$(e^{-1/3 t} y)' = e^{-4/3 t}$

$e^{-1/3 t} y = -3/4\, e^{-4/3 t} + c$

$\Rightarrow y(t) = e^{1/3 t} [-3/4\, e^{-4/3 t} + c]$

$= -3/4\, e^{-t} + c e^{t/3}$

$y(0) = -3/4 + c = a$

$\Rightarrow c = a + 3/4$

$\Rightarrow y(t) = -3/4\, e^{-t} (a + 3/4) e^{t/3}$ ⊠

Monday, January 26

recall: method of integrating factors

- $y' + P(x)y = Q(x)$
- $\mu = e^{\int P(x)\,dx}$
- $y = 1/\mu \left[\int \mu(x)\, Q(x)\, dx\right]$

example:

- $(1+t^2)y' + 4ty = 1/(1+t^2)^2$, $y(0)=1$

$= y' + 4t/(1+t^2)\, y = 1/(1+t^2)^3$

$\Rightarrow P(t) = 4t/(1+t^2)$, $Q(t) = 1 - 2/(1+t^2)$

$\mu(t) = e^{\int P(t)\,dt}$

$= e^{\int 4t/(1+t^2)\,dt}$

let $u = 1+t^2 \Rightarrow du = 2t\,dt \Rightarrow dt = \frac{du}{2t}$

$\Rightarrow e^{2\int du/u} = e^{2\ln|1+t^2|}$

$= (1+t^2)^2 = 1 + 2t^2 + t^4$

- $y'(1+t^2)^2 + y \cdot 4t(1+t^2) = 1/(1+t^2)$

$[y \cdot (1+t^2)^2]' = 1/(1+t^2)$

$y \cdot (1+t^2)^2 = \int 1/(1+t^2)\, dt$

$= \tan^{-1}(t) + C$

$\Rightarrow \quad y(t) = \tan^{-1}(t)(1+t^2)^{-2} + C(1+t^2)^{-2}$

- $y(0) = \cancelto{0}{\tan^{-1}(0)}(1+0^2)^{-2} + C(1+0^2)^{-2}$

$1 = C$

$\Rightarrow y(t) = \tan^{-1}(t)(1+t^2)^{-2} + (1+t^2)^{-2}$

$= (1+t^2)^{-2}(\tan^{-1}(t)+1)$ ⊠

example:

- $ty' - y = t^2e^{-t}$

$= y' - 1/t\, y = te^{-t}$

$\Rightarrow P(t) = -1/t$, $Q(t) = te^{-t}$

- $u(t) = e^{\int P(t)\,dt} = e^{\int -1/t\,dt}$

$= e^{-\ln|t|} = 1/t$

- $y' \cdot 1/t + y(-1/t^2) = e^{-t}$

$(y \cdot 1/t)' = e^{-t}$

$\Rightarrow y \cdot 1/t = \int e^{-t}\, dt$

$= -e^{-t} + C$

$\Rightarrow y(t) = -te^{-t} + Ct$ ⊠

example:

- $y' - 1/3\, y = e^{-t}$, $y(0) = a$

$\Rightarrow P(t) = -1/3$, $Q(t) = e^{-t}$

$\Rightarrow u(t) = e^{\int P(t)\,dt} = e^{\int -1/3\,dt}$

$= e^{-1/3\, t}$

$\Rightarrow y'e^{-t/3} + y(-1/3\, e^{-1/3\, t}) = e^{-4t/3}$

$(y\, e^{-t/3})' = e^{-4t/3}$

$\Rightarrow y e^{-t/3} = \int e^{-4t/3}\, dt$

$= -3/4\, e^{-4t/3} + C$

$\Rightarrow y(t) = -3/4\, e^{-t} + Ce^{t/3}$

- $y(0) = 3/4\, \cancelto{1}{e^{-0}} + C\cancelto{1}{e^{0/3}}$

$a = 3/4 + C$

$\Rightarrow C = a - 3/4$

$\Rightarrow y(t) = -3/4\, e^{-t} + (a - 3/4)e^{t/3}$ ⊠

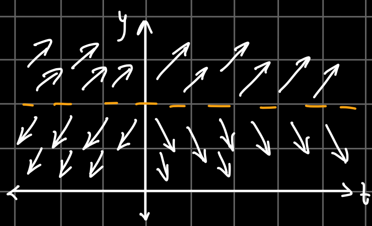
- $\lim_{t\to\infty} y(t) = -3/4\, \cancelto{0}{e^{-\infty}} + (a-3/4)e^{t/3}$

$= (a-3/4)e^{t/3}$

only need to check coeff. since $e^{t/3} > 0\ \forall t$

- if $a - 3/4 = 0 \Rightarrow a = 3/4 \Rightarrow \lim_{t\to\infty} y(t) = 0$
- if $a - 3/4 > 0 \Rightarrow a > 3/4 \Rightarrow \lim_{t\to\infty} y(t) = \infty$
- if $a - 3/4 < 0 \Rightarrow a < 3/4 \Rightarrow \lim_{t\to\infty} y(t) = -\infty$

- $\lim_{t\to-\infty} y(t) = -3/4\, e^{-(-\infty)} + (a-3/4)\cancelto{0}{e^{-\infty/3}} = -\infty$



chapter 2.2 : separable equations

- Let $dy/dx = f(x,y)$

$\Rightarrow dy/dx - f(x,y) = 0$

- then let $M(x,y) = -f(x,y)$ and $N(x,y) = 1$

$\Rightarrow dy/dx - f(x,y)$

$= \underbrace{M(x,y)}_{-f(x,y)} + \underbrace{N(x,y)}_{1}\, dy/dx = 0$

$\Rightarrow \underbrace{M(x,y)\,dx}_{\text{function of } x} + \underbrace{N(x,y)\,dy}_{\text{function of } y} = 0$

$\Rightarrow m(x)\,dx + N(y)\,dy = 0$ ← separation of variables

example: solve $dy/dx = x^2/(1-y^2)$ ← non-linear

- $dy/dx = x^2/(1-y^2)$

$\Rightarrow (1-y^2)\,dy = x^2\,dx$

$\Rightarrow \int (1-y^2)\,dy = \int x^2\,dx$

$y - \frac{1}{3}y^3 + C_1 = \frac{1}{3}x^3 + C_2$

$\Rightarrow 3y - y^3 = x^3 + K$ — $3(C_2 - C_1)$

$3y - y^3 - x^3 = K$ $\boxtimes$

example: solve $dy/dx = \dfrac{3x^2+4x+2}{2(y-1)}$, $y(0) = -1$

$\Rightarrow (2y-2)\,dy = (3x^2+4x+2)\,dx$

$\int (2y-2)\,dy = \int (3x^2+4x+2)\,dx$

$y^2 - 2y + C_1 = x^3 + 2x^2 + 2x + C_2$

$\Rightarrow y^2 - 2y = x^3 + 2x^2 + 2x + K$

$(y-1)^2 = x^3 + 2x^2 + 2x + K$

$\Rightarrow y(x) = 1 \pm \sqrt{x^3 + 2x^2 + 2x + K}$

- $y(0) = 1 \pm \sqrt{K}$

$-1 = 1 \pm \sqrt{K}$

$-2 = \pm\sqrt{K} \Rightarrow K = 4$

$\Rightarrow y(x) = 1 \pm \sqrt{x^3 + 2x^2 + 2x + 4}$

$= 1 \pm \sqrt{x^2(x+2) + 2(x+2)}$

$= 1 \pm \sqrt{(x+2)(x^2+2)}$ for $x \geq -2$

example: $dy/dx = \dfrac{4x - x^3}{4+y^3}$, $y(0) = 1$

$\Rightarrow (4+y^3)\,dy = (4x - x^3)\,dx$

$\int (4+y^3)\,dy = \int (4x - x^3)\,dx$

$\frac{1}{4}y^4 + 4y + C_1 = 2x^2 - \frac{1}{4}x^4 + C_2$

$y^4 + 16y = 8x^2 - x^4 + K$

- $y(0) \Rightarrow y^4 + 16(y) = K$

$1 + 16 = K$

$\Rightarrow K = 17$

$\Rightarrow y^4 + 16y = 8x^2 - x^4 + 17$ $\boxtimes$

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chapter 2.3 : modeling with differential equations

recall: growth / decay functions

- $dP/dt = KP$

$\Rightarrow dP/P = K\,dt$

$\int dP/P = \int K\,dt$

$\ln|P| = Kt + C$

$\Rightarrow P(t) = e^{Kt+C} = e^{Kt} \cdot e^{C}$

$P(t) = Ce^{Kt}$ $\boxtimes$

$K > 0 \Rightarrow$ growth → population growth, viruses

$K < 0 \Rightarrow$ decay → radioactivity → half life : $t_{1/2}$

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definition: doubling time (t_D)

- if t is doubled, t_D → doubling the amount of P_0

$P(t) = P_0 e^{Kt}$

$\Rightarrow 2P_0 = P_0 e^{Kt_D}$

$\Rightarrow 2 = e^{Kt_D}$

$\ln(2) = Kt_D$

$\Rightarrow t_D = \frac{1}{K}\ln(2)$ $\boxtimes$

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definition: half life

- this is the time needed to have half of the initial amount.
- $t_{1/2} = \frac{1}{K}\ln|1/2|$

$= \frac{1}{K}(\cancelto{0}{\ln(1)} - \ln(2))$

$= -\frac{1}{K}\ln 2$ $\boxtimes$

Friday, January 30

example: a radioactive material is reduced to 1/3 after 10 years. find its half-life.

- $P(t) = P_0 e^{kt}$

$\frac{1}{3}\cancel{P_0} = \cancel{P_0} e^{k(10)}$

$\frac{1}{3} = e^{10k}$

$\ln(\frac{1}{3}) = \cancel{\ln} e^{10k}$

$\ln(\frac{1}{3}) = 10k$

$\Rightarrow k = -\ln 3/10$

$\Rightarrow P(t) = P_0 e^{-\ln 3/10 \cdot t}$

- $\frac{1}{2}\cancel{P_0} = \cancel{P_0} e^{-\ln 3/10\, t}$

$-\ln 2 = -\ln 3/10 \cdot t$

$\Rightarrow t = 10\ln(2/3)$ ☒

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example: if the interest rate is 8%, compounded anually, find the doubling time.

- $P = P_0 e^{kt}$

$2\cancel{P_0} = \cancel{P_0} e^{0.08t}$

$2 = e^{0.08t}$

$\ln 2 = 0.08t$

$\Rightarrow t = \ln 2/0.08$

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example: start an IRA account at age 25. suppose you initially deposit \$3000 and you deposit \$2000 yearly at 8% APY, compounded continuously. Find value after 40 years.

- $S(t)$: IRA amount after t years
- $dS/dt = 2000 + 0.08S$, $S(0) = 3000$

$S' - 0.08S = 2000$

$\Rightarrow P(t) = -0.08$

$\mu(t) = e^{\int P(t)\,dt} = e^{\int 0.08\,dt}$

$= e^{-0.08t}$

$\Rightarrow S'e^{-0.08t} + S(-0.08e^{-0.08t}) = 2000e^{-0.08t}$

$(Se^{-0.08t})' = 2000e^{-0.08t}$

$\Rightarrow Se^{-0.08t} = 2000\int e^{-0.08t}\,dt$

$= 2000\, e^{-0.08} \cdot \frac{1}{0.08} + C$

$\Rightarrow S(t) = -25\,000 + Ce^{0.08t}$

- $S(0) = -25\,000 + C\cancel{e^0}$

$3000 = -25\,000 + C \Rightarrow C = 28\,000$

$\Rightarrow S(t) = -25000 + 28\,000e^{0.08t}$

$S(40) = -25{,}000 + 28{,}000e^{0.08(40)} = \$611{,}911$ ☒

example: a home buyer can pay \$800/month on mortgage payments at an annual interest rate of r (compounded anually). The mortgage term is 20 years. determine the maximum amount this buyer can afford to borrow. calculate the amount for (a) $r = 5\%$ and (b) $r = 9\%$.

- $dP/dt = rP - 800(12)$, $P(20) = 0$

$= rP - 9600$

$P' - rP = 9600$

- $\mu(t) = e^{\int -r\,dt} = e^{-rt}$
- $P'(e^{-rt}) + P(-re^{-rt}) = 9600e^{-rt}$

$Pe^{-rt} = 9600\int e^{-rt}\,dt$

$= -9600/r\, e^{-rt} + C$

$\Rightarrow P(t) = 9600/r + Ce^{rt}$

- $P(20) = 9600/r + Ce^{20r}$

$0 = 9600/r + Ce^{20r}$

$C = -1/r\; 9600e^{-20r}$

$\Rightarrow P(t) = 1/r\; 9600 - 1/r\; 9600e^{-20r}e^{rt}$

$= 1/r\; 9600(1 - e^{r(t-20)})$

(a) $P(t=0; r=0.05) = \frac{1}{0.05} \cdot 9600(1 - e^{0.05(-20)})$

$= \$121{,}347.$ ☒

(b) $P(t=0; r=0.09) = 1/0.09 \cdot 9600(1 - e^{0.09(-20)})$

$= \$89{,}035$ ☒

Monday, February 2

example:

- @ $t=0$, a tank containing Q_0 lb of salt dissolved in 100 gal of H_2O. Assume that water containing 1/4 lb of salt per gal is entering the tank at a rate of r gal/min. At the same time, the well mixed mixture is draining from the tank at the same rate.
- Find the amount of salt @ any time. $t > 0$

- Let $Q(t)$ be the amount of salt in the tank @ time t.
- dQ/dt = in rate − out rate

lb/min $= \frac{1}{4}\,\text{lb}/\cancel{g} \cdot r\,\cancel{g}/\text{min} - \frac{Q(t)}{100}\,\text{lb}/\cancel{g} \cdot r\,\cancel{g}/\text{min}$

$= \frac{1}{4}r - \frac{1}{100}Qr$

- $dQ/dt + r/100\, Q = \frac{1}{4}r$
- $\mu(t) = e^{\int P(t)\,dt} = e^{\int r/100\,dt} = e^{rt/100}$
- $Q'e^{rt/100} + Q(r/100 \cdot e^{rt/100}) = e^{rt/100} \cdot \frac{1}{4}r$

$(Qe^{rt/100}) = \frac{1}{4}r\int e^{rt/100}\,dt$

$= \frac{1}{\cancel{4}}\cancel{r} \cdot \frac{100}{\cancel{r}} \cdot e^{rt/100} + C$

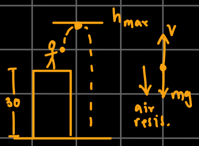
$\Rightarrow Q(t) = 25 + Ce^{-rt/100}$

$Q(0) = 25 + C \Rightarrow Q_0 - 25 = C$

$\Rightarrow Q(t) = 25 + (Q_0 - 25)e^{-rt/100}$ ☒

example:

- A ball with mass $1/2$ kg is thrown up with initial velocity 10 m/s from the roof of a building (30m high). Assume air resistance is $|v|/20$. Find the maximum above the ground the ball reaches.



- $\Sigma f = ma$

$-air - mg = ma$

$-v/20 - 1/2 \cdot 9.8 = 1/2\, dv/dt$

$1/2\, dv/dt + 1/20\, v = -1/2 \cdot 9.8$

$dv/dt + 1/10\, v = -9.8$

- $\mu(t) = e^{\int P(t)\,dt} = e^{\int 1/10\,dt} = e^{t/10}$
- $v'(e^{t/10}) + v(1/10 \cdot e^{t/10}) = -9.8 e^{t/10}$

$v e^{t/10} = -9.8 \int e^{t/10}\,dt$

$\Rightarrow v(t) = -98\, e^{t/10} + c$

$v(t) = -98 + ce^{-t/10}$

- $v(0) = -98 + c$

$10 = -98 + c \Rightarrow c = 108$

$\Rightarrow v(t) = -98 + 108 e^{-t/10}$ ☒

- @ h_{max}, $v(t) = 0$

$\Rightarrow 0 = -98 + 108 e^{-t/10}$

$98 = 108 e^{-t/10}$

$98/108 = e^{-t/10}$

$\ln 98/108 = -t/10$

$\Rightarrow t_{h_{max}} = -10 \ln(98/108) \approx 0.97$ s

- Recall, $v(t) = -98 + 108 e^{-t/10}$

$dh/dt = -98 + 108 e^{-t/10}$

$\Rightarrow h(t) = \int -98 + 108 e^{-t/10}\,dt$

$= -98t - 1080 e^{-t/10} + c$

- $h(0) = -1080 + c$

$30 = -1080 + c \Rightarrow c = 1110$ m

$\Rightarrow h(t) = -98t - 1080 e^{-t/10} + 1110$

- Then,

$h_{max} = h(0.97)$

$= -98(0.97) - 1080 e^{-0.97/10} + 1110$

$= 84.8$ m ☒

Wednesday, February 4

chapter 2.4: differences between linear & non-linear equations

- For a linear equation:

$y' + P(t)y = Q(t)$, $y(t_0) = y_0$,

we have the following theorem.

thm: linear existence & uniqueness

- if $P(t)$ and $Q(t)$ are continuous and bounded on an open interval (a,b) containing t_0, then, a unique solution exists on that interval.

- For a non-linear equation:

$y' = f(t,y)$, $y(t_0) = y_0$,

we have the following theorem

thm: non-linear existence & uniqueness

- if $f(t,y)$ and $\partial f/\partial y$ are continuous and bounded on a rectangle $(\alpha < t < \beta,\ a < y < b)$ containing (t_0, y_0), then $\exists$ an open interval around t_0 contained in (α, β) where the solution exists and is unique.

---

example: Find the interval where the solution can be defined.

$ty' + y = t^3$, $y(-1) = 3$

$y' + 1/t\, y = t^2$

using linearity

- t^2 is cts $\forall t$

$1/t$ is cts $\forall t$, $t \neq 0$

$\Rightarrow$ solution interval $= (-\infty, 0) \cup (0, \infty)$

- since $(-1,3) \in (-\infty, 0)$, the above interval contains a unique solution.

using non-linearity

- $y' = f(t,y)$

$\Rightarrow f(t,y) = t^2 - 1/t\, y$

- $f(t,y)$ is cts $\forall t$, $t \neq 0$. → guarantees existence

- $\partial f/\partial y = -1/t$

$\partial f/\partial y$ is cts $\forall t$, $t \neq 0$ → guarantees uniqueness

---

example: Find solution interval

- $(t-3)y' + \ln(t)\, y = 2$, $y(1) = 2$
- $y' + \frac{\ln(t)}{t-3} y = \frac{2}{t-3}$

using linearity:

- $P(t) = \frac{\ln(t)}{t-3}$ is cts $\forall t$, $t>0$, $t \neq 3$ $\Rightarrow$ solution exists

  $Q(t) = \frac{2}{t-3}$ is cts $\forall t$, $t \neq 3$ $\Rightarrow$ solution is unique

$\Rightarrow$ interval $= (0,3) \cup (3,\infty)$

- $1 \in (0,3)$ $\therefore$ solution is defined and unique.

using non-linearity:

- $y' = \frac{2}{t-3} - \left(\frac{\ln t}{t-3}\right) y = f(t,y)$

  $f(t,y)$ is cts $\forall t$, $t>0$, $t \neq 3$ $\Rightarrow$ solution exists
- $\partial f/\partial y = \frac{-\ln t}{3-t}$

  $\partial f/\partial y$ is cts $\forall t$, $t>0$, $t \neq 3$ $\Rightarrow$ solution is unique

$\Rightarrow$ interval $= (0,3) \cup (3,\infty)$

- $1 \in (0,3)$ $\therefore$ solution is defined and unique.

---

example: $dy/dx = \sqrt{x-y}$, $y(2) = 2$ (1). Find solution interval.

- (1) is non-linear so

$$dy/dx = f(x,y) = \sqrt{x-y}$$

- For continuity,

$$x - y \geq 0$$

- However, since we take an open interval

$$x - y > 0 \Rightarrow x > y$$

- From the initial condition

$$x = 2 \not> y = 2$$

so (1) is discontinuous at $(2,2)$ and we can't guarantee the existence of a solution.

Friday, February 6

example: $dy/dx = 2x^2y^2$, $y(1) = -1$. Find solution interval.

- The given equation is non-linear so,

$$dy/dx = f(x,y) = 2x^2y^2$$

- $f(x,y)$ is continuous $\forall x,y \in \mathbb{R}$ so a solution exists on $(1,-1)$.
- Now,

$$\partial f/\partial y = 4x^2y$$

- $\partial f/\partial y$ is continuous $\forall x,y \in \mathbb{R}$ so the solution is unique.

---

example: $dy/dx = x \ln y$, $y(1) = 1$

- non-linear so,

$$y' = f(x,y) = x \ln y$$

- $f(x,y)$ is cts $\forall x$, $y > 0$.
- $(1,1) \in \{(x,y) \mid y > 0\}$ so a solution exists.
- $\partial f/\partial y = x/y$ is cts $\forall x$, $y \neq 0$ so the solution is unique.

---

example: $dy/dx = \sqrt[3]{y}$, $y(0) = 1$

- non-linear so,

$$y' = f(x,y) = y^{1/3}$$

- $f(x,y)$ is cts $\forall y$ so a sol. exists.
- $\partial f/\partial y = \frac{1}{3}y^{-2/3}$ is cts $\forall y > 0$.
- $(0,1) \in \{(x,y) \mid y > 0\}$ so the sol. is unique.

---

example: $dy/dx = \ln(1+y^2)$; $y(0) = 0$

- $y' = f(x,y) = \ln(1+y^2)$

  $\Rightarrow 1+y^2 > 0 \Rightarrow y^2 > -1$, holds $\forall y$

$\Rightarrow f(x,y)$ is cts $\forall y$ so sol. exists.

- $\partial f/\partial y = \frac{2y}{1+y^2} \Rightarrow 1+y^2 \neq 0 \Rightarrow y^2 \neq -1$, holds $\forall y$

  so sol. is unique.

Wednesday, February 11

chapter 2.6: exact equations & integrating factors

recall:

- consider $f(x,y)$
- $\partial f/\partial x = f_x$, $\partial^2 f/\partial x^2 = f_{xx}$, $\partial f/\partial x\partial y = f_{xy}$
- $\partial f/\partial y = f_y$, $\partial^2 f/\partial y^2 = f_{yy}$, $\partial f/\partial y\partial x = f_{yx}$
- now, let $x = g(t)$, $y = h(t)$

$$\Rightarrow f(x,y) = f(x(t), y(t))$$
$$\Rightarrow df/dt = \partial f/\partial x \cdot dx/dt + \partial f/\partial y \cdot dy/dt$$
$$= f_x \cdot x' + f_y \cdot y' \quad (1)$$

---

example: let $f(x,y) = x^2y^2 + e^x$, $x(t) = t^2$, $y(t) = e^t$. Find df/dt.

- $df/dt = f_x \cdot x' + f_y \cdot y'$

$$= (2xy^2 + e^x)\cdot 2t + (2x^2y)e^t$$
$$= (2t^2e^{2t} + e^{t^2})2t + (2t^4 e^t)e^t$$
$$= (2t^2e^{2t} + e^{t^2})2t + 2t^4e^{2t}$$

---

definition: exact equations

- An equation of the form

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is exact if $\exists$ a function $F(x,y)$ such that

$$F_x = M$$
$$F_y = N$$

- we should also satisfy

$$F_{xy} = F_{yx}$$
$$\Rightarrow M_y = N_x$$

- To solve, we find $F(x,y)$ and set it equal to c.

---

example: $(2x+3y) + (x-2y)y' = 0$. Check for exactness.

- $M(x,y) = 2x+3y$

  $N(x,y) = x-2y$
- WTS: $M_y = N_x$
- $M_y = 3$, $N_x = 1$

  $M_y \neq N_y$ so the equation is not exact

---

example: $(2x+y) + (x+2y)y' = 0$; $y(1) = 1$. Check for exactness.

- $M(x,y) = 2x+y$, $N(x,y) = x+2y$

  $M_y = 1 = 1 = N_x \Rightarrow$ we have exactness.
- Now, solving:

$$F_x = M = 2x+y, \quad F_y = N = x+2y$$
$$\Rightarrow F(x,y) = \int F_x\,dx = x^2 + xy + g(y)$$
$$\Rightarrow F_y = \partial/\partial y\, F(x,y)$$
$$= x + g'(y)$$
$$= x + 2y \quad \Rightarrow g(y) = \int 2y\,dy = y^2$$
$$\Rightarrow F(x,y) = x^2 + xy + y^2 = c$$

- using $y(1) = 1$

$$c = 1 + 1 + 1 = 3$$
$$\Rightarrow 3 = x^2 + xy + y^2$$

---

example: $(xy^2 + bx^2y) + (x+y)x^2y' = 0$. Find b and solve.

- $M = xy^2 + bx^2y$, $N = (x+y)x^2 = x^3 + x^2y$
- $M_y = 2xy + bx^2 = 3x^2 + 2xy = N_x$

$\Rightarrow b = 3$.

- Now,

$$F_x = M = xy^2 + 3x^2y, \quad F_y = N = x^3 + x^2y$$
$$\Rightarrow F(x,y) = \tfrac{1}{2}x^2y^2 + x^3y + g(y)$$
$$F_y = x^2y + x^3 + g'(y)$$
$$= x^2y + x^3$$
$$\Rightarrow g'(y) = 0 \Rightarrow g(y) = 0$$
$$\Rightarrow F(x,y) = \tfrac{1}{2}x^2y^2 + x^3y$$
$$\Rightarrow c = \tfrac{1}{2}x^2y^2 + x^3y \quad \boxtimes$$

---

topic: integrating factor

- we use it to switch a non-exact ODE into an exact one.
- Recall,

$$M(x,y)\,dx + N(x,y)\,dy = 0 \quad (1)$$

- Multiplying (1) by $\mu(x,y)$

$$\mu M\,dx + \mu N\,dy = 0 \quad (2)$$

- Equation (2) is exact if

$$(\mu M)_y = (\mu N)_x \quad (2a)$$

- Using the product rule on (2a):

$$\mu_y M + \mu M_y = \mu_x N + \mu N_x \quad (3)$$
$$M\mu_y - N\mu_x + (M_y - N_x)\mu = 0 \quad (4)$$

- If a function μ is found and satisfies (4), then (2) will be exact.
- Assuming μ is a function of x only, the (4) becomes

$$0 - N\mu_x + (M_y - N_x)\mu = 0$$
$$\Rightarrow \mu_x = \left(\frac{M_y - N_x}{N}\right)\mu$$
$$d\mu/dx = \left(\frac{M_y - N_x}{N}\right)\mu$$
$$d\mu/dx - \left(\frac{M_y - N_x}{N}\right)\mu = 0$$
$$\sim d\mu/dx - P(x)\mu = Q(x) \quad (5)$$

example: $(3xy+y^2)+(x^2+xy)y' = 0$; $y(1)=2$ (1)

$\Rightarrow M(x,y) = 3xy+y^2 \Rightarrow M_y = 3x+2y$

$N(x,y) = x^2+xy \Rightarrow N_x = 2x+y$

$\Rightarrow M_y \neq N_x$ so the equation is not exact.

$\Rightarrow \mu' = \left(\frac{M_y - N_x}{N}\right)\mu$

$= \left(\frac{3x+2y-(2x+y)}{x^2+xy}\right)\mu = \left(\frac{x+y}{x^2+xy}\right)\mu$

$\frac{d\mu}{dx} = \left(\frac{\cancel{x+y}}{x(\cancel{x+y})}\right)\mu = \frac{\mu}{x}$

$\Rightarrow d\mu/\mu = dx/x$

$\Rightarrow \ln|\mu| = \ln|x|$

$\mu = x$

- Now, using μ:

$x(3xy+y^2)+x(x^2+xy)y'=0$

$\Rightarrow (3x^2y+xy^2)+(x^3+x^2y)y'=0$ (2)

- $F_x = M(x,y) = 3x^2+xy^2 \; dx \Rightarrow M_y = 3x^2+2xy$

$F_y = N(x,y) = x^3+x^2y \; dy \Rightarrow N_x = 3x^2+2xy$

- $M_y = N_x \Rightarrow$ (2) is exact
- Now,

$F(x,y) = \int F_x \, dx$

$= \int 3x^2+xy^2 \, dx$

$= x^3y + \frac{1}{2}x^2y^2 + h(y)$

- $F_y(x,y) = x^3 + x^2y + h'(y)$
- $F_y \overset{set}{=} N(x,y)$

$\cancel{x^3}+\cancel{x^2y}+h'(y) = \cancel{x^3}+\cancel{x^2y}$

$\Rightarrow h'(y)=0 \Rightarrow h(y)=0$

- Now

$F(x,y) = x^3y+\frac{1}{2}x^2y^2+0$

$\Rightarrow C = x^3y+\frac{1}{2}x^2y^2$

- Finally

$y(1)=2$

$\Rightarrow C = (1)(2)+\frac{1}{2}(1)(4)$

$= 2+2 = 4$

$\Rightarrow 4 = x^3y+\frac{1}{2}x^2y^2$ ☒

Monday, February 16

chapter 3: 2nd order diff. eqs

chapter 3.1: homogeneous diff. eqs with constant coeffs.

- 1st order diff eq: $dy/dx = F(x,y)$
- 2nd order diff eq: $d^2y/dx^2 = F(x,y,y')$
- nth order diff eq: $d^ny/dx^n = F(x,y,y',\dots,y^{(n-1)})$

definition: general forms of 2nd order linear diff eqs.

$a_2(x)y''+a_1(x)y'+a_0(x)y = f(x)$

$\Rightarrow y'' + \frac{a_1(x)}{a_2(x)}y' + \frac{a_0(x)}{a_2(x)}y = \frac{f(x)}{a_2(x)}$

$\Rightarrow y''+P(x)y'+Q(x)y = R(x)$ (1)

2nd order linear diff eq. in standard form

Wednesday, February 18

recall:

- $d^2y/dx^2 + P(x)\,dy/dx + Q(x)y = R(x)$ (1)

is a 2nd order linear diff. eq in standard form.

---

definition: homogeneous and coefficient characterization

- If $R(x)=0$ in (1), then (1) is called homogeneous.
- If $P(x)$ and $Q(x)$ are constants, then we call (1) a constant coeff. ODE. Otherwise, it is a variable coeff. ODE

---

example: $3y''-5y'+6y=0$

$\Rightarrow y''-\frac{5}{3}y'+2y=0$

homogeneous and constant coeff.

---

example: $2y''-xy=3x-5$

$\Rightarrow y''-\frac{x}{2}y = \frac{3}{2}x-\frac{5}{2}$

nonhomogeneous and variable coeff.

---

example: $6x^2y''+4xy'+y=0$

$\Rightarrow y''+\frac{2}{3x}y'+\frac{1}{6x^2}y=0$

homogeneous and variable coeff.

topic: solving linear homo. 2nd order ODE's with constant coeff.

- Let $y(t) = e^{rt} \Rightarrow y'(t) = re^{rt} \Rightarrow y''(t) = r^2 e^{rt}$

---

example: $y''+3y'+2y=0 \quad ; \ y(0)=1, y'(2)$

- Let $y(t)=e^{rt}$
- substituting,

$$y''+3y'+2y=0$$
$$\Rightarrow r^2e^{rt}+3(re^{rt})+2(e^{rt})=0$$
$$e^{rt}(r^2+3r+2)=0$$
$$\Rightarrow \cancel{e^{rt}\neq 0} \quad \text{or} \quad (r^2+3r+2)=0$$
$$\Rightarrow (r+2)(r+1)=0$$
$$\Rightarrow r_1=-2, \ r_2=-1$$
$$\Rightarrow y(t) = c_1e^{r_1t}+c_2e^{r_2t}$$
$$= c_1e^{-2t}+c_2e^{-t} \quad \text{(linearly independent)}$$
$$\Rightarrow y'(t) = -2c_1e^{2t}-c_2e^{-t}$$

- Finding c_i,
  - $y(0) = c_1e^0+c_2e^0$

    $1 = c_1+c_2 \Rightarrow c_1 = 1-c_2$
  - $y'(0) = -2c_1^0 - c_2^0$

    $2 = -2c_1 - c_2$
- Solving by elimination

$$1 = c_1+\cancel{c_2}$$
$$+ \quad 2 = -2c_1-\cancel{c_2}$$
$$3 = -c_1 \Rightarrow c_1 = -3$$
$$\Rightarrow c_2 = 4$$

$\Rightarrow y(t) = 4e^{-t}-3e^{2t}$ ☒

---

example: $y''+2y'-3y=0 \ ; \ y(0)=5, \ y'(1)=7$

- Using $y(t)=e^{rt}$

$$y''+2y'-3y=0$$
$$\Rightarrow r^2+2r-3=0$$
$$(r-1)(r+3)=0$$
$$\Rightarrow r=1, \ r=-3$$

$\Rightarrow y(t) = c_1e^t+c_2e^{-3t}$

$y'(t) = c_1e^t-3c_2e^{-3t}$

- Using initial conditions

$y(0) = c_1e^0 - ce^{-0}$

$5 = c_1 - c_2 \quad \Rightarrow c_1 = 5+c_2$

$y'(1) = c_1e^1 - 3c_2e^{-3(1)}$

$7 = c_1e - 3c_2e^{-3}$

- Solving

$7 = (5+c_2)e-3c_2e^{-3}$

$7 = 5e+c_2e-3c_2e^{-3} \quad \Rightarrow \quad 7 = 5e+c_2(e-3e^{-3})$

$\Rightarrow c_2 = 7-5e/e-3e^{-3} \Rightarrow c_1 = 5+\left(7-5e/e-3e^{-3}\right)$

$\Rightarrow y(t) = 5+\left(7-5e/e-3e^{-3}\right)e^t + \left(7-5e/e-3e^{-3}\right)e^{-3t}$ ☒

---

chapter 3.2: solutions of linear homo. ODE. the Wronskian.

thm: existence and uniqueness thm to 2nd order linear diff. eq.

- consider IVP:

$$y''+P(t)y'+Q(t)y = R(t) \ ; \ y(t_0)=y_0, \ y'(t_0)=\tilde{y}_0$$

- if $P(t)$, $Q(t)$ and $R(t)$ are cts. and bounded on an open interval I containing t_0, $\exists!$ sol. $y(t)$ of this eq. defined on I

Friday, February 20

recall:

- we are interested in solving 2nd order ODE that is
  1. linear
  2. homogeneous
  3. with constant coeff.
- let's consider:

$$y''+P(t)y'+Q(t)y=0$$

idea:

- we seek two solutions y_1 and y_2 that are linearly independent and no sol. can be expressed in terms of sols. this set is referred to as the fundamental set of solutions.
- then, the general sol. is a linear combination of the two sols:

$$y = c_1y_1+c_2y_2$$

---

example: $(t^2-3t)y''+ty'+(t+3)y = e^t \ ; \ y(1)=2, \ y'(1)=1$

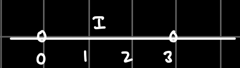
Find the largest interval I where a solution is valid

$$\Rightarrow y'' + \underbrace{\left(\frac{t}{t^2-3t}\right)}_{P(t)}y' + \underbrace{\left(\frac{t+3}{t^2-3t}\right)}_{Q(t)}y = \underbrace{\frac{e^t}{t^2-3t}}_{R(t)}$$

- For continuity, $t^2-3t \neq 0 \Rightarrow t(t-3)=0$

  $\Rightarrow t\neq 0, \ t\neq 3$
- using initial conditions

$\Rightarrow I = (0,3)$



---

definition: Wronskian

- given two functions $f(t)$ and $g(t)$, the Wronskian is defined as

$$W(f(t),g(t)) = \begin{vmatrix} f(t) & g(t) \\ f'(t) & g'(t) \end{vmatrix} = f(t)g'(t)-g(t)f'(t)$$

- If $W(f(t),g(t))=0$, then $f(t)$ and $g(t)$ are linearly dependent. Otherwise, they are linearly independent.

example: check if the following are linearly independent.

(a) $f(t) = e^{t}$, $g(t) = e^{-t}$

- $f'(t) = e^{t}$, $g'(t) = -e^{-t}$
- $W(f'(t), g'(t)) = e^{t} \cdot -e^{-t} - e^{t} \cdot e^{-t}$
  $= -e^{0} - e^{0} = -1 - 1$
  $= 2 \neq 0$

$\Rightarrow$ e^{t} & e^{-t} are LI.

(b) $f(t) = \sin(t)$, $g(t) = \cos(t)$

- $f'(t) = -\cos(t)$, $g'(t) = \sin(t)$
- $W(f(t), g(t)) = \sin^2(t) + \cos^2(t) = 1 \neq 0$

$\Rightarrow$ $\sin(t)$ & $\cos(t)$ are LI.

(c) $f(t) = t+1$, $g(t) = 4t+4$

$g(t) = 4f(t) \Rightarrow f(t)$ & $g(t)$ are LD.

---

remark:

- Suppose that $y_1(t)$ and $y_2(t)$ are sols. of $y'' + P(t)y' + Q(t)y = 0$. Then, if

  $W(y_1(t), y_2(t)) \neq 0$

  then $y_1(t)$ and $y_2(t)$ are LI and form a fundamental set of solutions.

---

example: show that $y_1(t) = t^{1/2}$, $y_2(t) = t^{-1}$ form a fundamental set of sols. to $2t^2y'' + 3ty' - y = 0$

① $y_1'(t) = \frac{1}{2}t^{-1/2}$, $y_1''(t) = -\frac{1}{4}t^{-3/2}$

$\Rightarrow 2t^2\left(-\frac{1}{4}t^{-3/2}\right) + 3t\left(\frac{1}{2}t^{-1/2}\right) - t^{1/2} \overset{?}{=} 0$

$-\frac{1}{2}t^{1/2} + \frac{3}{2}t^{1/2} - t^{1/2} \overset{?}{=} 0$

$0 = 0$ ✓

② $y_2'(t) = -t^{-2}$, $y_2''(t) = 2t^{-3}$

$2t^2(2t^{-3}) + 3t(-t^{-2}) - t^{-1} \overset{?}{=} 0$

$4t^{-1} - 3t^{-1} - t^{-1} \overset{?}{=} 0$

$0 = 0$

- Now,

  $W(y_1(t), y_2(t)) = t^{1/2} \cdot -t^{-2} - t^{-1} \cdot \frac{1}{2}t^{-1/2}$
  $= -t^{-3/2} - \frac{1}{2}t^{-3/2}$
  $= -\frac{3}{2}t^{-3/2} \neq 0$

$\Rightarrow$ $y_1(t)$ & $y_2(t)$ are LI are form a fundamental set of solns.

---

thm: Abel's theorem

- y_1 and y_2 are LI solutions to

  $y'' + P(t)y' + Q(t)y = 0$

  on an open interval (I). Then,

  $W(y_1, y_2)(t) = Ce^{-\int P(t)\,dt}$

  for some constant C depends on y_1 and y_2 but independent on t or I.

Monday, February 23

example: Find $W(y_1, y_2)$ without solving

$t^2y'' - t(t+2)y' + (t+2)y = 0$

$\Rightarrow y'' - \left(\frac{t+2}{t}\right)y' + \left(\frac{t+2}{t}\right)y = 0$

$y'' - (1 + 2/t)y' + (1 + 2/t)y = 0$

$\Rightarrow P(t) = -(1 + 2/t)$

$\Rightarrow W(y_1, y_2)(t) = Ce^{-\int P(t)\,dt} = Ce^{\int 1 + 2/t\,dt}$
$= Ce^{t + 2\ln t} = Ce^{t} \cdot \cancel{e}^{\cancel{\ln} t^2}$
$= Ce^{t} t^2 \neq 0$

$\Rightarrow$ y_1 & y_2 are LI.

---

example: If y_1 & y_2 are two solns. of $ty'' + 2y' + e^{t}y = 0$ and $W(y_1, y_2)(1) = 2$. Find $W(y_1, y_2)(5)$.

$\Rightarrow y'' + \left(\frac{2}{t}\right)y' + e^{t}(y) = 0$

- $W(y_1, y_2)(t) = ce^{-\int \frac{2}{t}\,dt} = ce^{-2\ln t}$
  $= c\cancel{e}^{\cancel{\ln} t^{-2}} = c \cdot 1/t^2 \neq 0$
- $W(y_1, y_2)(1) = c \cdot 1/1^2$
  $2 = c$

$\Rightarrow W(y_1, y_2)(t) = 2/t^2$

$\Rightarrow W(y_1, y_2)(5) = 2/25$ ⊠

---

example: if $W(f, g) = 3e^{4t}$, where $f(t) = e^{2t}$. Find $g(t)$.

- Recall, $W(f, g)(t) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = \begin{vmatrix} e^{2t} & g \\ 2e^{2t} & g' \end{vmatrix}$

  $3e^{4t} = e^{2t}g'(t) - 2e^{2t} \cdot g(t)$

  $3e^{2t} = g'(t) - 2g(t)$ → 1st order linear diff. eq.

$\Rightarrow \mu = e^{\int P(t)\,dt} = e^{\int -2\,dt}$
$= e^{-2t}$

$\Rightarrow e^{-2t}g'(t) - 2(e^{-2t})g(t) = 3$

$[e^{-2t} \cdot g(t)]' = 3$

$e^{-2t} \cdot g(t) = \int 3\,dt$

$e^{-2t} \cdot g(t) = 3t + c$

$\Rightarrow g(t) = 3t \cdot e^{2t} + ce^{2t}$

example: Solve and find a general solution by Abel's theorem.

$y'' + 2y' + y = 0$

$\Rightarrow r^2 + 2r + 1 = 0 \Rightarrow (r+1)^2 = 0$

$\Rightarrow r_{1,2} = -1$

$\Rightarrow y_1 = c_1 e^{-t}$

- $W(y_1, y_2) = Ce^{-\int 2\,dt} = Ce^{-2t}$

$\Rightarrow Ce^{-2t} = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} c_1e^{-t} & y_2 \\ -c_1e^{-t} & y_2' \end{vmatrix}$

$Ce^{-2t} = c_1 e^{-t} \cdot y_2' + c_1 e^{-t} y_2$

$\Rightarrow e^{-2t} = e^{-t} y_2' + e^{-t} y_2$

$e^{-t} = y_2' + y_2$

$\Rightarrow \mu = e^{\int P(t)\,dt} = e^{t}$

$\Rightarrow e^{t} y_2' + e^{t} y_2 = 1$

$(e^{t} y)' = t + c$

$\Rightarrow y(t) = te^{-t} + \underbrace{ce^{-t}}_{y_1}$

$\Rightarrow y_2 = te^{-2t}$

$\Rightarrow y(t) = c_1 e^{-t} + c_2 t e^{-t}$ ⊠

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chapter 3.3: complex roots

- consider $y'' + y = 0$. Find a general solution.
- $r^2 + 1 = 0 \Rightarrow r = \pm\sqrt{-1}$

$= \pm i$

- first: find a func. such as $y_1 = \sin(t)$ & $y_2 = \cos(t)$
- second: find the wronskian
  - $W(y_1, y_2) = \begin{vmatrix} \sin(t) & \cos(t) \\ \cos(t) & -\sin(t) \end{vmatrix} = -\sin^2 t - \cos^2 t$

$= -1 \neq 0$

$\Rightarrow y_1, y_2$ are LI

$\Rightarrow y(t) = c_1 \sin(t) + c_2 \cos(t)$

Wednesday, February 25

- consider $ay'' + by' + cy = 0$

$\Rightarrow ar^2 + br + c = 0$

$\Rightarrow r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

- if $b^2 - 4ac < 0$ then,

$r_1 = \frac{-b}{2a} + \frac{\sqrt{b^2-4ac}}{2a} = \frac{-b}{2a} + i \cdot \frac{\sqrt{-b^2+4ac}}{2a}$

$r_2 = \frac{-b}{2a} - i \frac{\sqrt{-b^2+4ac}}{2a}$

- Let $\lambda = \frac{-b}{2a}$ and $\mu = \frac{\sqrt{-b+4ac}}{2a}$

$\Rightarrow r_1 = \lambda + \mu i$, $r_2 = \lambda - \mu i$

$\Rightarrow y_1 = e^{(\lambda + \mu i)t}$; $y_2 = e^{(\lambda - \mu i)t}$

$y_1 = e^{\lambda t} \cdot e^{\mu t i}$; $y_2 = e^{\lambda t} e^{-\mu i t}$ — $e^{i\theta} = \cos\theta + i\sin\theta$

$y_1 = e^{\lambda t}[\cos(\mu t) + i\sin(\mu t)]$; $y_2 = e^{\lambda t}[\cos(\mu t) + i\sin(-\mu t)]$

$\Rightarrow y_1 + y_2 = 2e^{\lambda t}\cos(\mu t) \Rightarrow y_1^* = \frac{1}{2}(y_1 + y_2) = e^{\lambda t}\cos(\mu t)$

$y_1 - y_2 = 2i \cdot e^{\lambda t}\sin(\mu t) \Rightarrow y_2^* = \frac{1}{2}(y_1 - y_2) = e^{\lambda t}\sin(\mu t)$

- $W(y_1^*, y_2^*) = \mu e^{2\lambda t} \neq 0$

$\Rightarrow y_1^*$ & y_2^* are LI. $\Rightarrow y(t) = c_1 e^{\lambda t}\cos(\mu t) + c_2 e^{\lambda t}\sin(\mu t)$

Friday, February 27

recall: complex roots

- $r = \lambda \pm i\mu$

$\Rightarrow y(t) = e^{\lambda t}[c_1\cos(\mu t) + c_2\sin(\mu t)]$

example: $y'' + 4y = 0$; $y(\frac{\pi}{6}) = 0$, $y'(\frac{\pi}{6}) = 1$

$\Rightarrow r^2 + 4 = 0$

perfect (uniform) oscillation

$\Rightarrow r = \pm 2i \Rightarrow \lambda = 0, \mu = 2$

$\Rightarrow y(t) = \cancel{e^{0t}}^{\,1}[c_1\cos(2t) + c_2\sin(2t)]$

$y(t) = c_1\cos(2t) + c_2\sin(2t)$

$y'(t) = -2c_1\sin(2t) + 2c_2\cos(2t)$

- Using initial conditions:

$y(\frac{\pi}{6}) = c_1\cos(\frac{\pi}{3}) + c_2\sin(\frac{\pi}{3})$

$0 = \frac{1}{2}c_1 + \frac{\sqrt{3}}{2}c_2 \Rightarrow c_1 + \sqrt{3}c_2 = 0$

$y'(\frac{\pi}{6}) = -2c_1\sin(\frac{\pi}{3}) + 2c_2\cos(\frac{\pi}{3})$

$1 = -\sqrt{3}c_1 + c_2 \Rightarrow 3c_1 - \sqrt{3}c_2 = -\sqrt{3}$

- Solving by elimination:

$c_1 = \frac{-\sqrt{3}}{4} \Rightarrow c_2 = \frac{1}{4}$

$\Rightarrow y(t) = -\frac{\sqrt{3}}{4}\cos(2t) + \frac{1}{4}\sin(2t)$ ⊠

---

example: $y'' + 2y' + 101y = 0$; $y(0) = 1$, $y'(0) = 0$

$\Rightarrow r^2 + 2r + 101 = 0$

$\Rightarrow r = \frac{-2 \pm \sqrt{4 - 404}}{2} = -1 \pm \sqrt{-100}$

decay oscillation

$= -1 \pm 10i \Rightarrow \lambda = -1, \mu = 10$

$\Rightarrow y(t) = e^{-t}[c_1\cos(10t) + c_2\sin(10t)]$

$y'(t) = e^{-t}[c_1\cos(10t) + c_2\sin(10t)] - e^{-t}[10c_1\sin(10t) + 10c_2\cos(10t)]$

$\Rightarrow y(0) = e^{-0}[c_1\cos(0) + \cancel{c_2\sin(0)}^{\,0}]$

$1 = c_1$

$y'(0) = -e^{-0}[c_1\cos(0) + \cancel{c_2\sin(0)}^{\,0}] - e^{-t}[\cancel{10c_1\sin(0)}^{\,0} + 10c_2\cos(0)]$

$0 = c_1 - 10c_2 \Rightarrow c_2 = \frac{1}{10}$

$\Rightarrow y(t) = e^{-t}[\cos(10t) + \frac{1}{10}\sin(10t)]$ ⊠

---

example: $y'' - y' + 81.25y = 0$. Find $\lim_{t\to\infty} y(t)$

$\Rightarrow r^2 - r + 2025 = 0$

growth oscillation

$\Rightarrow r = \frac{1 \pm \sqrt{1 - 4 \cdot 2025}}{2} = \frac{1}{2} \pm 9i \Rightarrow \lambda = \frac{1}{2}, \mu = 9$

$\Rightarrow y(t) = e^{t/2}[c_1\cos(9t) + c_2\sin(9t)]$

- $\lim_{t\to\infty} y(t) = \infty$ ⊠

chapter 3.4: Repeated roots; Reduction of order

example: $y''+4y'+4y=0$

$\Rightarrow r^2+4r+4=0$

$\Rightarrow (r+2)^2=0 \Rightarrow r=-2$ (repeated)

$\Rightarrow y_1 = c_1e^{-2t}$

- Now, $W(y_1,y_2) = ce^{-\int p(t)dt} = ce^{-4t}$ (1)

$W(y_1,y_2) = \begin{vmatrix} c_1e^{-2t} & y_2 \\ -2c_1e^{-2t} & y_2' \end{vmatrix}$ (2)

$= c_1e^{-2t} y_2' + 2c_1e^{-2t}y_2$

- Equating (1) and (2):

$ce^{-4t} = c_1\cancel{e^{-2t}}y_2' + 2c_1\cancel{e^{-2t}}y_2$

$ce^{-2t} = c_1y_2' + 2c_1y_2$

$\Rightarrow e^{-2t} = y_2' + 2y_2$

- Now, $\mu = e^{\int p(t)dt} = e^{2t}$

$\Rightarrow 1 = (e^{2t}y_2)'$

$\Rightarrow y_2 = te^{-2t}$

- Then, $y(t) = c_1e^{-2t} + c_2te^{-2t}$ ⊠

Monday, March 2

recall:

- $y''+4y'+4y=0$ (1)
- $y(t) = c_1e^{-2t}+c_2te^{-2t}$
- $(r+2)^2=0 \Rightarrow r=-2$

$\Rightarrow y_1(t) = e^{-2t}$

- let $y_2(t) = v(t)e^{-2t}$

$\Rightarrow y_2' = v'(t)e^{-2t} + v(t)(-2)e^{-2t}$

$= v'(t)e^{-2t} - 2v(t)e^{-2t}$

$y_2'' = (v''(t)e^{-2t} - 2v'(t)e^{-2t}) - (2v'(t)e^{-2t} - 4v(t)e^{-2t})$

$= v''(t)e^{-2t} - 2v'(t)e^{-2t} - 2v'(t)e^{-2t} + 4v(t)e^{-2t}$

- sub into (1)

$v''(t)e^{-2t} - \cancel{2v'(t)e^{-2t}} - \cancel{2v'(t)e^{-2t}} + \cancel{4v'(t)e^{-2t}} + \cancel{4v'(t)e^{-2t}} - \cancel{8v(t)e^{-2t}} + \cancel{4v(t)e^{-2t}} = 0$

$\Rightarrow v''(t)e^{-2t} = 0$

$\Rightarrow v''(t)=0 \Rightarrow v'(t)=c_1 \Rightarrow v(t) = c_1t+c_2$???

$\Rightarrow y_2(t) = v(t)e^{-2t}$

$= (c_1t+c_2)e^{-2t}$

$= c_1te^{-2t}+c_2e^{-2t}$

- Since $y_1(t) = e^{-2t}$ & $y_2(t) = c_1te^{-2t}+c_2e^{-2t}$ we don't repeat it

$\Rightarrow y_2(t) = c_1te^{-2t}$

- let $c_1=1$

$\Rightarrow y_2(t) = te^{-2t}$

$\Rightarrow y(t) = c_1e^{-2t}+c_2te^{-2t}$ ⊠

example: $y''-2y'+y=0$; $y(0)=2$, $y'(0)=1$

$\Rightarrow r^2-2r+r=0$

$(r-1)^2=0 \Rightarrow r_1=r_2=1$

$\Rightarrow y(t) = c_1e^t + c_2te^t$

$y'(t) = c_1e^t + (c_2e^t + c_2te^t)$

- $y(0) = c_1 = 2$
- $y'(0) = c_1+c_2 = 1 \Rightarrow c_2=-1$

$\Rightarrow y(t) = 2e^t - te^t$ ⊠

summary: for $ay''+by'+cy=0$

1. $r_1 \neq r_2 \Rightarrow y(t) = c_1e^{r_1t}+c_2e^{r_2t}$
2. $r_1=r_2 \in \mathbb{R} \Rightarrow y(t) = c_1e^{rt}+c_2te^{rt}$
3. $r=\lambda+i\mu \in \mathbb{C} \Rightarrow y(t) = e^{\lambda t}[c_1\sin(\mu t)+c_2\cos(\mu t)]$

example: $2t^2y''+3ty'-y=0$, $t>0$

given $y_1=\frac{1}{t}$. Find a second L.I solution.

$\Rightarrow y'' + \left(\frac{3}{2t}\right)y' - \left(\frac{1}{2t^2}\right)y = 0$ → variable coefficient 2nd order ODE

- L.I $\Rightarrow W(y_1,y_2) = ce^{-\int p(t)dt} = ce^{-\frac{3}{2}\int\frac{1}{t}dt}$

$= ce^{-\frac{3}{2}\ln(t)} = ce^{\ln(t)^{-3/2}}$

$= ct^{-3/2} \neq 0$

- $W(y_1,y_2) = y_1y_2' - y_2y_1' = \frac{1}{t}y_2' - y_2\left(-\frac{1}{t^2}\right)$

$t^{-3/2} = \frac{1}{t}y_2' + \frac{1}{t^2}y_2$

$t^{-1/2} = y_2' + \frac{1}{t}y$

- $\mu = e^{\int p(t)dt} = e^{\ln(t)} = t$

$\Rightarrow t^{1/2} = (ty_2)'$

$\frac{2}{3}t^{3/2} + c = ty_2$

$\Rightarrow y_2(t) = \frac{2}{3}t^{1/2} + \frac{c}{t}$ ⊠

Wednesday, March 4

chapter 3.5: Non-homogeneous diff eq's; method of undetermined coefficients

- we solved 2nd order homog. linear diff. eq's with const. coeffs.
- Now, we need to solve non. homog. linear DE's such as:

$y''+p(t)y'+q(t)y = g(t)$ (1)

1. Find solution for the homog. eq using previous methods and call the solution y_h.
   assume a solution in the form of $g(t) \notin y_h$
2. Find particular solution Y for (1) by method of undetermined coefficients.
3. The general solution $y(t) = y_h + Y$

example: $y'' - 3y' - 4y = 3e^{2t}$

1. Find y_h

$\Rightarrow r^2 - 3r - 4 = 0 \Rightarrow (r-4)(r+1) \Rightarrow r_1 = 4,\ r_2 = -1$

$\Rightarrow y_h = c_1 e^{4t} + c_2 e^{-t}$

2. Find Y

- Assume a solution in the form $3e^{2t}$

$\Rightarrow Y = Ae^{2t} \Rightarrow Y' = 2Ae^{2t} \Rightarrow Y'' = 4Ae^{2t}$

- $Y'' - 3Y' - 4Y = 3e^{2t}$

$\cancel{4Ae^{2t}} - 6Ae^{2t} - \cancel{4Ae^{2t}} = 3e^{2t}$

$\Rightarrow A = -\frac{1}{2}$

$\Rightarrow Y = -\frac{1}{2}e^{2t}$

$\Rightarrow y(t) = c_1 e^{4t} + c_2 e^{-t} - \frac{1}{2}e^{2t}$ ⊠

---

example: $y'' - 3y' - 4y = 2e^{-t}$

- $y_h = c_1 e^{4t} + c_2 e^{-t}$
- $Y = Ae^{-t} \Rightarrow Y' = -Ae^{-t} \Rightarrow Y'' = Ae^{-t}$
- $Ae^{-t} + 3Ae^{-t} - 4Ae^{-t} = 2e^{-t}$

$0 = 2$ ✗

- Doesn't work since $Y = Ae^{-t} \subset y_h$
- So let $Y = Ate^{-t} \Rightarrow Y' = Ae^{-t} - Ate^{-t} \Rightarrow Y'' = -Ae^{-t} - Ae^{-t} + Ate^{-t}$

$= Ate^{-t} - 2Ae^{-t}$

$\Rightarrow Ate^{-t} - 2Ae^{-t} - 3Ae^{-t} + 3Ate^{-t} - 4Ate^{-t} = 2e^{-t}$

$\cancel{At} - 2A - 3A + \cancel{3At} - \cancel{4At} = 2$

$5A = 2 \Rightarrow A = \frac{2}{5}$

$\Rightarrow Y = \frac{2}{5}te^{-t}$

$\Rightarrow y(t) = c_1 e^{4t} + c_2 e^{-t} + \frac{2}{5}te^{-t}$ ⊠

---

summary: exponentials

- if $g(t) = \alpha e^{\alpha t}$, then, the form of the particular solution (Y) depends on r_1 and r_2 (roots of characteristic eq.).

case	form of particular solution Y
$r_1 \neq r_2 \neq \alpha$	$Y = Ae^{\alpha t}$
$r_1 = \alpha$ or $r_2 = \alpha$	$Y = Ate^{\alpha t}$
$r_1 = r_2 = \alpha$	$Y = At^2 e^{\alpha t}$

---

example: $y'' - 3y' - 4y = 3t^2 + 2$

- $y_h = c_1 e^{4t} + c_2 e^{-t}$
- Let $Y = At^2 + Bt + C \Rightarrow Y' = 2At + B \Rightarrow Y'' = 2A$

$\Rightarrow 2A - 6At - 3B - 4At^2 - 4Bt - 4C = 3t^2 + 2$

$-4At^2 - (6A + 4B)t - (4A + 3B + 4C) = 3t^2 + 2$

$\Rightarrow A = -3/4$

- $6(-3/4) + 4B = 0 \Rightarrow B = \frac{9}{8}$
- $-4(-3/4) - 3(\frac{9}{8}) - 4C = 2 \Rightarrow C = (3 - \frac{27}{8} - 2)\frac{1}{4} = \frac{55}{32}$

$\Rightarrow Y = \frac{-3}{4}t^2 + \frac{9}{8}t + \frac{55}{32}$

$\Rightarrow y(t) = c_1 e^{4t} + c_2 e^{-2t} - \frac{3}{4}t^2 + \frac{9}{8}t + \frac{55}{32}$ ⊠

Friday, March 6

example: $y'' - 3y' = 3t^2 + 2$

$\Rightarrow r^2 - 3r = 0 \Rightarrow r(r-3) = 0$

$\Rightarrow r_1 = 0,\ r_2 = 3$

$\Rightarrow y_c(t) = c_1 + c_2 e^{3t}$

- Let $Y = t(At^2 + Bt + C) = At^3 + Bt^2 + Ct$

$\Rightarrow Y' = 3At^2 + 2Bt + C \Rightarrow Y'' = 6At + 2B$

$\Rightarrow (6At + 2B) - 3(3At^2 + 2Bt + C) = 3t^2 + 2$

$(-9A)t^2 + (6A - 6B)t + (2B - 3C) = 3t^2 + 2$

$\Rightarrow -9A = 3 \Rightarrow A = -1/3$

$\Rightarrow 6(-1/3) - 6B = 0 \Rightarrow -6B = 2 \Rightarrow B = -1/3$

$\Rightarrow 2(-1/3) - 3C = 2 \Rightarrow -2/3 - 2 = 3C \Rightarrow C = -8/9$

$\Rightarrow Y = -1/3\,t^3 - 1/3\,t^2 - 8/9\,t$

$\Rightarrow y(t) = c_1 + c_2 e^{3t} - 1/3\,t^3 - 1/3\,t^2 - 8/9\,t$ ⊠

---

summary: polynomials

- if $g(t)$ is a polynomial of degree n,

$g(t) = a_n t^n + \dots + a_1 t + a_0$

then, the particular solution for $ay'' + by' + cy = g(t)$, $a \neq 0$ depends on b and c.

case	$Y(t)$
$c \neq 0$	$Y(t) = P_n(t) = A_n t^n + \dots + A_1 t + A_0$
$c = 0, b \neq 0$	$Y(t) = t \cdot P_n(t)$
$c = b = 0$	$Y(t) = t^2 \cdot P_n(t)$

---

example: $y'' - 3y' - 4y = \sin(t)$

- $r^2 - 3r - 4 = 0 \Rightarrow (r-4)(r+1) = 0$

$\Rightarrow y_c(t) = c_1 e^{4t} + c_2 e^{-t}$

- let $Y = A\sin(t) + B\cos(t)$

$Y' = A\cos(t) - B\sin(t)$, $Y'' = -A\sin(t) - B\cos(t)$

$\Rightarrow -A\sin(t) - B\cos(t) - 3A\cos(t) + 3B\sin(t) - 4A\sin(t) - 4B\cos(t) = \sin(t)$

$\sin(t)[-A + 3B - 4A] + \cos(t)[-B - 3A - 4B] = \sin(t)$

$\Rightarrow (-5A + 3B = 1)5$

$\Rightarrow + (-3A - 5B = 0)3$

$-34A = 5 \Rightarrow A = -5/34,\ B = 3/34$

$\Rightarrow Y = -5/34\sin(t) - 3/34\cos(t)$

$\Rightarrow y(t) = c_1 e^{4t} + c_2 e^{-t} - 5/34\sin(t) - 3/34\cos(t)$ ⊠

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summary: trigonometric

1. if $g(t) = \alpha\cos(t)$ or $g(t) = \alpha\sin(t)$, then

$Y(t) = A\sin(t) + B\cos(t)$

2. if $g(t) = A\sin(t) + B\cos(t)$, then

$Y(t) = A\sin(t) + B\cos(t)$

example: $y'' + y = \sin(t)$

$\Rightarrow r^2 + 1 = 0 \Rightarrow r = \pm\sqrt{-1} = \pm i$

$\Rightarrow \lambda = 0, \mu = 1$

$\Rightarrow y_c(t) = e^{0t}[c_1 \cos(t) + c_2 \sin(t)]$

$= c_1 \cos(t) + c_2 \sin(t)$

- let $Y = At\sin(t) + Bt\cos(t)$

$\Rightarrow Y' = [A\sin(t) + At\cos(t)] + [B\cos t - Bt\sin(t)]$

$= t[A\cos(t) - B\sin(t)] + (A\sin(t) + B\cos(t)]$

$\Rightarrow Y'' = A\cos(t) - B\sin(t) + t[-A\sin(t) - B\cos(t)] +$

$A\sin(t) - B\cos(t)$

$= t[-A\sin(t) - B\cos(t)] + (A-B)\sin(t) + (A-B)\cos(t)$

$\Rightarrow t[\cancel{-A\sin(t)} - \cancel{B\cos(t)}] + (A-B)\sin(t) + (A+B)\cos(t)$

$+ t[\cancel{A\sin(t)} + \cancel{B\cos(t)}] = \sin(t)$

$\Rightarrow A - B = 1 \Rightarrow 2A = 1 \Rightarrow A = 1/2 \Rightarrow B = -1/2$

$A + B = 0 \Rightarrow A = -B$

$\Rightarrow Y = \frac{1}{2}t[\sin(t) - \cos(t)]$

$\Rightarrow y(t) = c_1\cos(t) + c_2\sin(t) + \frac{1}{2}t[\sin(t) - \cos(t)]$ ⊠

---

summary: trigonometric

- if $g(t) = a\sin(\alpha t) + b\cos(\alpha t)$, then the particular solution depends on r_1 & r_2.

case	$Y(t)$
$r_1 \neq r_2 \neq \alpha i$	$Y(t) = A\sin(\alpha t) + B\cos(\alpha t)$
$r_1, r_2 = \pm\alpha i$	$Y(t) = t[A\sin(\alpha t) + B\cos(\alpha t)]$

---

example: $y'' - 3y' - 4y = te^t$

- $r^2 - 3r - 4 = 0 \Rightarrow (r-4)(r+1) = 0$

$\Rightarrow r_1 = 4, r_2 = -1, \alpha = 1$

$\Rightarrow y_c(t) = c_1 e^{4t} + c_2 e^{-t}$

- $g(t) = te^t$

$\Rightarrow Y = (At+B)(e^t) \Rightarrow Y' = Ae^t + (At+B)e^t = (At+B+A)e^t$

$\Rightarrow Y'' = 2Ae^t + (At+B)e^t$

$= e^t(At + B + 2A)$

- Substituting,

$e^t(At+B+2A) - 3Ae^t - 3e^t(At+B) - 4e^t(At+B) = te^t$

$e^t(At + B + 2A - 3A - 3At - 3B - 4At - 4B) = te^t$

$\Rightarrow t(A - 3A - 4A) = t$

$\Rightarrow -6A = 1 \Rightarrow A = -1/6$

$\Rightarrow (B + 2A - 3A - 3B - 4B) = 0$

$-A - 6B = 0 \Rightarrow 6B = -A = 1/36$

$\Rightarrow Y = (1/36 - 1/6t)e^t$

$\Rightarrow y(t) = c_1 e^{4t} + c_2 e^{-t} + (1/36 - 1/6t)e^t$ ⊠

---

example: $y'' - 3y' - 4y = te^{-t}$

- $r_1 = 4, r_2 = -1, \alpha = -1$

- $y_c(t) = c_1 e^{4t} + c_2 e^{-t}$

- $r_2 = \alpha = 1 \Rightarrow Y = (At+B)te^{-t} = e^{-t}(At^2 + Bt)$

$\Rightarrow Y' = -e^{-t}(At^2 + Bt) + e^{-t}(2At + B)$

$= e^{-t}(2At + B - At^2 - Bt) = e^{-t}[-At^2 + (2A-B)t + B]$

$\Rightarrow Y'' = -e^{-t}[-At^2 + (2A+B)t + B] + e^{-t}[-2At + 2A - B]$

$= e^{-t}[At^2 + (4A+B)t + (2B-2A)]$

- Substituting

- $\cancel{e^{-t}}[At^2 + (4A+B)t + (2B-2A) + 3At^2 - 6At - 3Bt - 3B - 4At^2 - 4Bt) = t\cancel{e^{-t}}$

$\Rightarrow A = -\frac{1}{10}, B = -\frac{1}{25}$

$\Rightarrow Y = (-\frac{1}{10}t^2 - \frac{1}{25}t)e^{-t}$

$\Rightarrow y(t) = c_1 e^{4t} + c_2 e^{-t} + (-\frac{1}{10}t^2 - \frac{1}{25}t)e^{-t}$ ⊠

---

summary:

1. $g(t) = P_n(t) = A_n t^n + \ldots + A_1 t + A_0$
   - $r_1, r_2 \neq \alpha \Rightarrow Y(t) = P_n(t)e^{\alpha t}$
   - $r_1, r_2 = \alpha \Rightarrow Y(t) = P_n(t) \cdot te^{\alpha t}$
   - $r_1 = r_2 = \alpha \Rightarrow Y(t) = P_n(t) \cdot t^2 e^{\alpha t}$

2. $g(t) = e^{\alpha t}[a\cos(\beta t) + b\sin(\beta t)]$
   - $r_1, r_2 \neq \alpha + i\beta \Rightarrow Y(t) = e^{\alpha t}[A\cos(\beta t) + B\sin(\beta t)]$
   - $r_1, r_2 = \alpha + i\beta \Rightarrow Y(t) = te^{\alpha t}[A\cos(\beta t) + B\sin(\beta t)]$

3. $g(t) = P_n(t)e^{\alpha t}[a\cos(\beta t) + b\sin(\beta t)]$
   - $r_1, r_2 \neq \alpha + i\beta = e^{\alpha t}[P_n(t)\cos(\beta t) + P_n(t)\sin(\beta t)]$
   - $r_1, r_2 = \alpha + i\beta = te^{\alpha t}[P_n(t)\cos(\beta t) + P_n(t)\sin(\beta t)]$

4. $g(t) = g_1(t) + \ldots + g_n(t)$

   $\Rightarrow Y(t) = Y_1(t) + \ldots + Y_n(t)$

---

each frequency ($2t$ & t) gets a sin, cos pair

example: $y'' - 5y' + 6y = e^t\cos(2t) + e^{2t}(3t+4)\sin(t)$

- $r^2 - 5r + 6 = 0 \Rightarrow (r-3)(r-2)$

$\Rightarrow r_1 = 2, r_2 = 3 \Rightarrow y_c(t) = c_1 e^{2t} + c_2 e^{3t}$

- $g_1(t) = e^t\cos(2t) \Rightarrow \alpha = 1, \beta = 2$

$\Rightarrow Y_1(t) = e^t[A\cos(2t) + B\sin(2t)]$

- $g_2(t) = e^{2t}(3t+4)\sin(t)$

$\Rightarrow Y_2(t) = e^{2t}[(At+B)\cos(t) + (Ct+D)\sin(t)]$

$\Rightarrow Y(t) = e^t[A\cos(2t) + B\sin(2t)] + e^{2t}[(Ct+D)\cos(t) + (Et+F)\sin(t)]$

$\Rightarrow y(t) = y_c(t) + Y(t)$ ⊠

Monday, March 16

example: $y'' - 2y' + y = 1$; $y(0) = 2$, $y'(0) = 7$ (1)

- Homogeneous: $y'' - 2y' + y = 0$

$\Rightarrow r^2 - 2r + 1 = 0 \Rightarrow (r-1)^2 = 0 \Rightarrow r_{1,2} = 1$

$\Rightarrow y_h = c_1 e^t + c_2 t e^t$ (2)

- Particular solution: $y_p = At^2 + Bt + C$ (3)

$\Rightarrow y'_p = 2At + B \Rightarrow y''_p = 2A$

- Substituting,

$2A - 2(2At + B) + At^2 + Bt + C = 1$

$At^2 + (B - 4A)t + (2A - 2B + C) = 1$

$\Rightarrow 2A - 2B + C = 1$

$\Rightarrow B - 4A = 0 \Rightarrow B = 4A$

$\Rightarrow A = 0 = B \Rightarrow C = 1$

$\Rightarrow y_p = C = 1$

- general solution: $y(t) = c_1 e^t + c_2 t e^t + 1$ (4)

$y(0) = c_1 + 1$

$2 = c_1 + 1 \Rightarrow c_1 = 1$

$y'(t) = c_1 e^t + c_2 e^t + c_2 t e^t$

$y'(0) = c_1 + c_2$

$7 = 1 + c_2 \Rightarrow c_2 = 6$

$\Rightarrow y(t) = e^t + 6te^t + 1$ ⊠

---

example: $y'' + y = 1$; $y(0) = 2$, $y'(0) = 7$

- homogeneous: $y'' + y = 0$

$\Rightarrow r^2 + 1 = 0 \Rightarrow r = \pm\sqrt{-1} = \pm i$

$\Rightarrow \lambda = 0$, $\mu = 1$

$\Rightarrow y_c = e^{0t}[c_1 \cos(t) + c_2 \sin(t)]$

$= c_1 \cos(t) + c_2 \sin(t)$

- particular: let $y_p = C = 1$
- general: $y(t) = c_1 \cos(t) + c_2 \sin(t) + 1$
  - $y(0) = c_1 + 1 = 2 \Rightarrow c_1 = 1$
  - $y'(t) = -c_1 \sin(t) + c_2 \cos(t)$

  $y'(0) = c_2 = 7$

  $\Rightarrow y(t) = \cos(t) + 7\sin(t) + 1$ ⊠

---

example: $y'' - 7y' + 12y = 4e^{2x}$

- homogeneous: $y'' - 7y' + 12y = 0$

$\Rightarrow r^2 - 7r + 12 = 0 \Rightarrow (r-3)(r-4) = 0 \Rightarrow r = 3, 4$

$\Rightarrow y_H = c_1 e^{3x} + c_2 e^{4x}$

- particular: $\alpha = 2 \neq r \Rightarrow y_p = Ae^{2x}$

$\Rightarrow y'_p = 2Ae^{2x} \Rightarrow y''_p = 4Ae^{2x}$

- substituting,

$4Ae^{2x} - 14Ae^{2x} + 12Ae^{2x} = 4e^{2x}$

$2A = 4 \Rightarrow A = 2$

$\Rightarrow y_p = 2e^{2x}$

- general: $y(x) = c_1 e^{3x} + c_2 e^{4x} + 2e^{2x}$ ⊠

---

example: $y'' - 7y' + 12y = 6e^{4x}$

- homogeneous: $y_H = c_1 e^{2x} + c_2 e^{4x}$
- particular: $\alpha = 4 = r \Rightarrow y_p = Axe^{4x}$

$\Rightarrow y'_p = Ae^{4x} + 4Axe^{4x} \Rightarrow y''_p = 4Ae^{4x} + 4Ae^{4x} + 16Axe^{4x}$

$= 8Ae^{4x} + 16Axe^{x}$

- substituting,

$8A + 16Ax - 7(A + 4Ax) + 12Ax = 5$

$8A + \cancel{16Ax} - 7A + \cancel{4Ax} + \cancel{12Ax} = 5$

$A = 5$

$\Rightarrow y_p = 5xe^{4x}$

- general: $y(t) = c_1 e^{2x} + c_2 e^{4x} + 5xe^{4x}$ ⊠

---

example: $y'' - 3y' + 2y = e^{3x}(x^2 + 2x - 1)$

- homogeneous: $y'' - 3y' + 2y = 0$

$\Rightarrow r^2 - 3r + 2 = 0 \Rightarrow (r-2)(r-1) = 0 \Rightarrow r = 1, 2$

$\Rightarrow y_H = c_1 e^x + c_2 e^x$

- particular: $\alpha = 3 \neq r \Rightarrow y_p = e^{3x}(Ax^2 + Bx + C)$

$\Rightarrow y'_p = 3e^{3x}(Ax^2 + Bx + C) + e^{3x}(2Ax + B)$

$y''_p = 9e^{3x}(Ax^2 + Bx + C) + e^{3x}(2Ax + B) + 3e^{3x}(2Ax + B) + e^{3x}(2A)$

- substituting,

$9Ax^2 + 9Bx + 9C + 2Ax + B + 6Ax + 3B + 2A -$

$3(3Ax^2 + 3Bx + 3C + 2Ax + B) +$

$2(Ax^2 + Bx + C) = x^2 + 2x - 1$

- $\cancel{9Ax^2} - \cancel{9Ax^2} + 2Ax^2 + 8Ax + \cancel{9Bx} - 6Ax - \cancel{9Bx} + 2Bx +$

$2A + 4B + 3C - 3B - \cancel{9C} + \cancel{9C} = x^2 + 2x - 1$

$2Ax^2 + (2A + 2B)x + (2A + B + 3C) = x^2 + 2x - 1$

$\Rightarrow 2A = 1 \Rightarrow A = 1/2$

$\Rightarrow 3 - 2B = 2 \Rightarrow B = 1/2$

$\Rightarrow 1 + 1/2 + 3C = -1 \Rightarrow -3/2$

$\Rightarrow y_p = e^{3x}(\frac{1}{2}x^2 + \frac{1}{2}x - 3/2)$

- general: $y(x) = c_1 e^x + c_2 e^x + e^{3x}(\frac{1}{2}x^2 + \frac{1}{2}x - \frac{3}{2})$ ⊠

Friday, March 20

chapter 4: higher order diff. equations

§4.1: general theory of n^{th} order linear diff. eqs.

- the general n^{th} order linear diff. eq takes the form.

$$y^{(n)} + p_{n-1}(t)\, y^{(n-1)} + \dots + p_1(t)\, y' + p_0(t)\, y = g(t) \qquad (1)$$

- if all coefficients $(p_{n-1}(t), \dots, p_1(t), p_0(t))$ are constants, then (1) is a constant coeff. n^{th} order DE.
- if at least one of the coeff funcs is a func. of t, then (1) is said to be a variable coeff ODE.
- if $g(t) = 0$, then (1) is homogeneous.

---

theorem: existence and uniqueness

- if the coefficient functions, $p_0(t), p_1(t), \dots, p_{n-1}(t)$ are cts and bounded on a open interval I containing t_0, the (1) has a unique solution on I.
- Wronksian: $(n \times n)$ matrix
  - $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$
  - $W(y_1, \dots, y_n) = \begin{vmatrix} y_1 & \cdots & y_n \\ y_1' & \ddots & \vdots \\ y_1'' & & \vdots \\ \vdots & & \\ y_1^{(n-1)} & \cdots & \cdots \end{vmatrix}$
- $y(t) = y_H + y_p$ for non-homogeneous cases

---

§4.2: higher order homogeneous diff. eqs. with constant coefficients

- consider $a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = 0$

$\Rightarrow a_n r^n + a_{n-1} r^{n-1} + \dots + a_1 r + a_0 = 0$

- roots: n roots (count multiplicity)
  - $\Rightarrow (r - r_1)(r - r_2) \dots (r - r_n) = 0$
- solutions: n solutions

example: $y^{(4)} - 4y'' = 0$, $y(0) = 0, y'(0) = 0, y''(0) = 8, y^{(3)}(0) = 0$

$\Rightarrow r^4 - 4r^2 = 0 \Rightarrow r^2(r^2 - 4) = 0$

$\Rightarrow r = 0$ (mult. $= 2$), $r = \pm 2$

$\Rightarrow y(t) = c_1 e^{0t} + c_2 t e^{0t} + c_3 e^{2t} + c_4 e^{-2t}$

$= c_1 + c_2 t + c_3 e^{2t} + c_4 e^{-2t}$

$\Rightarrow y(0) = c_1 + c_2 + c_4 = 0$

- $y'(t) = c_2 + 2c_3 e^{2t} - 2c_4 e^{-2t}$

  $y'(0) = c_2 + 2r_3 - 2c_4 = 0$

- $y''(t) = 4c_3 e^{2t} + 4c_4 e^{-2t}$

  $y''(0) = 4c_3 + 4c_4 = 8$

- $y^{(3)}(t) = 8c_3 e^{2t} - 8c_4 e^{-2t}$

  $y^{(3)}(0) = 8c_3 - 8c_4 = 0$

$$\left.\begin{array}{l} \Rightarrow 16c_3 = 16 \Rightarrow c_3 = 1 \\ \Rightarrow 8c_3 - 8c_4 = 0 \Rightarrow c_4 = 1 \\ \Rightarrow c_2 = 2c_4 - 2c_3 = 0 \\ \Rightarrow c_1 = -c_4 - c_2 = -1 \end{array}\right\} y(t) = -1 + e^{2t} - e^{-2t} \quad \boxtimes$$

example: $y^{(4)} + 4y'' = 0$

$\Rightarrow r^4 + 4r^2 = 0 \Rightarrow r^2(r^2 + 4) = 0$

$r_{1,2} = 0$, $r_{3,4} = \pm 2i$

$\Rightarrow y(t) = c_1 + c_2 t + c_3 \cos(2t) + c_4 \sin(2t) \quad \boxtimes$

---

example: $y^{(3)} - y = 0$

$\Rightarrow r^3 - 1 = 0 = (r-1)(r^2 + r + 1) = 0$

$\Rightarrow r_1 = 1$

$- r_{2,3} = \dfrac{-1 \pm \sqrt{1-4}}{2} = -\dfrac{1}{2} \pm \dfrac{\sqrt{3}}{2} i$

$\Rightarrow y(t) = c_1 e^t + e^{-t/2}\left[c_2 \cos\left(\frac{\sqrt{3}}{2} t\right) + c_3 \sin\left(\frac{\sqrt{3}}{2} t\right)\right]$

---

example: $y^{(4)} + 8y'' + 16y = 0$

$\Rightarrow r^4 + 8r^2 + 16 = 0 \Rightarrow (r^2 + 4)^2 = 0$

$\Rightarrow r_{1,2,3,4} = \pm 2i$ (mult. 2)

$\Rightarrow y(t) = c_1 \cos(2t) + c_2 \sin(2t) + t[c_3 \cos(2t) + c_4 \sin(2t)]$

---

example: $y^{(3)} + 3y'' + 3y' + y = 0$

$\Rightarrow r^3 + 3r^2 + 3r + 1 = 0$

$$\begin{array}{r|l} & r^2 + 2r + 1 \\ \hline r+1 & r^3 + 3r^2 + 3r + 1 \\ & -(\cancel{r^3} + r^2) \\ \hline & \cancel{2r^2} + 3r \\ & -(\cancel{2r^2} + 2r) \\ \hline & r + 1 \end{array}$$

$\Rightarrow r^3 + 3r^2 + 3r + 1 = (r+1)(r^2 + 2r + 1)$

$= (r+1)(r+1)^2 = (r+1)^3$

$\Rightarrow r_{1,2,3} = -1$

$\Rightarrow y(t) = c_1 e^{-t} + c_2 t e^{-t} + c_3 t^2 e^{-t} \quad \boxtimes$

Monday, March 23

§ 4.3: method of undetermined coefficients

example: $y^{(3)} - 3y'' + 3y' - y = 4e^t$

- homo: $y^{(3)} - 3y'' + 3y' - y = 0$

$\Rightarrow r^3 - 3r^2 + 3r - 1 = 0$

$(r-1)(r^2 - 2r + 1) = 0$

$(r-1)(r-1)^2 = (r-1)^3$

$$\begin{array}{r|l} & r^2 - 2r + 1 \\ \hline r-1 & r^3 - 3r^2 + 3r - 1 \\ & -(\cancel{r^3} - r^2) \\ & \cancel{-2r^2} + 3r \\ & +(-\cancel{2r^2} - 2r) \\ & r - 1 \end{array}$$

$\Rightarrow r_{1,2,3} = 1$

$\Rightarrow y_H = c_1 e^t + c_2 t e^t + c_3 t^2 e^t$

- particular: $y_p = At^3 e^t$

$\Rightarrow y'_p = 3At^2e^t + At^3e^t = Ae^t(3t^2 + t^3)$

$y''_p = 6Ate^t + 3At^2e^t + 3At^2e^t + At^3e^t$

$= 6Ate^t + 6At^2e^t + At^3e^t$

$= Ae^t(6t + 6t^2 + t^3)$

$y^{(3)}_p = Ae^t(6t + 6t^2 + t^3) + Ae^t(6 + 12t + 3t^2)$

$= Ae^t(t^3 + 9t^2 + 18t + 6)$

$\Rightarrow y^{(3)} - 3y'' + 3y' - y = A\cancel{e^t}(\cancel{t^3} + \cancel{9t^2} + \cancel{18t} + 6) - 3A\cancel{e^t}(\cancel{6t} + \cancel{6t^2} + \cancel{t^3})$
$+ 3A\cancel{e^t}(\cancel{3t^2} + \cancel{t^3}) - \cancel{Ae^t}\cancel{t^3}$

$\Rightarrow 4 = 6A \Rightarrow A = \frac{2}{3}$

$\Rightarrow y_p = \frac{2}{3}t^3e^t$

- general: $y(t) = c_1e^t + c_2te^t + c_3t^2e^t + \frac{2}{3}t^3e^t$

---

example: $y^{(4)} + 2y'' + y = 3\sin(t) - 5\cos(t)$

- homo: $y^{(4)} + 2y'' + y = 0$

$\Rightarrow r^4 + 2r^2 + 1 = 0 \Rightarrow (r^2+1)(r^2+1) = 0$

$\Rightarrow (r^2+1)^2 = 0 \Rightarrow r^2 + 1 = 0$

$\Rightarrow r = \pm i$

$\Rightarrow y_H = c_1\cos(t) + c_2\sin(t) + c_3\cos(-t) + c_4\sin(-t)$

- $\cos(-t) = \cos(t)$ & $\sin(-t) = -\sin(t)$

$\Rightarrow y_H = c_1\cos(t) + c_2\sin(t)$

- particular: $y_p = t[A\sin(t) + B\cos(t)]$

$y'_p = At\cos t + A\sin t - Bt\sin t + B\cos t$

$= \sin(t)(A - Bt) + \cos(t)(B + At)$

$y''_p = \sin(t)(-2B - At) + \cos(t)(2A - Bt)$

$y'''_p = -\sin(t)(3A - Bt) + \cos(t)(B + At)$

$y^{(4)}_p = -t\sin(t) + \cos(t)(Bt - 2A)$

$\Rightarrow 3 = 2B \Rightarrow B = \frac{3}{2}$

$\Rightarrow -5 = 2A \Rightarrow A = -\frac{5}{2}$

$\Rightarrow y_p = t\left[\frac{3}{2}\cos(t) - \frac{5}{2}\sin(t)\right]$

- general: $y(t) = c_1\cos(t) + c_2\sin(t) + t\left[\frac{3}{2}\cos(t) - \frac{5}{2}\sin(t)\right]$ ⊠

Wednesday, March 25

§ 6.1: The Laplace Transform

definition:

- Let $f(t)$ be a function defined for $t \geq 0$. Then, the improper integral

$$\int_0^\infty f(t)e^{-st}\,dt = \lim_{a\to\infty}\int_0^a f(t)e^{-st}\,dt$$

is equivalent to

$$F(s) = \mathcal{L}\{f(t)\}$$

- This implies

$$\mathcal{L}\{f(t)\} = \lim_{a\to\infty}\int_0^a f(t)e^{-st}\,dt$$

where s is a real number called the parameter of the transform and provided that the improper integral converges.

---

example: $f(t) = 1$

$\Rightarrow \mathcal{L}\{f(t)\} = \mathcal{L}\{1\} = \lim_{a\to\infty}\int_0^a e^{-st}\,dt$

$= \lim_{a\to\infty}\left[\frac{e^{-st}}{-s}\right]_0^a = -\frac{1}{s}\lim_{a\to\infty}\left[e^{-st}\right]_0^a$

$= -\frac{1}{s}\lim_{a\to\infty}\left[e^{-as} - e^0\right] = -\frac{1}{s}\left[e^{-\infty} - 1\right] = \frac{1}{s}$ ⊠

---

example: $f(t) = e^{at}$

$\Rightarrow \mathcal{L}\{e^{at}\} = \lim_{b\to\infty}\int_0^b e^{at}e^{-st}\,dt = \lim_{b\to\infty}\int_0^\infty e^{t(a-s)}\,dt$

$= \frac{1}{a-s}\lim_{b\to\infty}\left[e^{t(a-s)}\right]_0^b = \frac{1}{a-s}\lim_{b\to\infty} e^{b(a-s)} - 1$

$= -\frac{1}{a-s}\left[1 - \lim_{b\to\infty} e^{b(a-s)}\right] = \frac{-1}{a-s}[1 - 0]$

$= \frac{1}{s-a}, \quad s > a$ ⊠

---

example: $f(t) = t$

$\Rightarrow \mathcal{L}\{t\} = \lim_{a\to\infty}\int_0^a t\cdot e^{-st}\,dt$

$$\begin{array}{ccc} t & & e^{-st} \\ 1 & + & e^{-st}/-s \\ 0 & - & e^{-st}/s^2 \end{array}$$

since $e^{-\infty}$ is dominant

$= \lim_{a\to\infty}\left[\frac{te^{-st}}{-s} - \frac{e^{-st}}{s^2}\right]_0^a = \lim_{a\to\infty}\left[\left(\cancel{\frac{ae^{-as}}{-s}} - \cancel{\frac{e^{-as}}{s^2}}\right) - \left(0 - \frac{1}{s^2}\right)\right]$

$= \frac{1}{s^2}$ ⊠

---

example: $f(t) = t^2$

$\Rightarrow \mathcal{L}\{t^2\} = \lim_{a\to\infty}\int_0^a t^2e^{-st}\,dt$

$$\begin{array}{ccc} t^2 & & e^{-st} \\ 2t & + & e^{-st}/-s \\ 2 & - & e^{-st}/s^2 \\ 0 & + & e^{-st}/-s^3 \end{array}$$

$= \lim_{a\to\infty} \frac{t^2e^{-st}}{-s} - \frac{2te^{-st}}{s^2} - \frac{2e^{-st}}{s^3} = \frac{2}{s^3}$ ⊠

example: $f(t) = t^3$

$\Rightarrow \mathcal{L}\{t^3\} = \lim_{a\to\infty} \int_0^a t^3 e^{-st}\, dt$

$$\begin{array}{ccc} & D & I \\ + & t^3 & e^{-st} \\ - & 3t^2 & e^{-st}/-s \\ + & 6t & e^{-st}/s^2 \\ - & 6 & e^{-st}/-s^3 \\ & 0 & e^{-st}/s^4 \end{array}$$

$= \lim_{a\to\infty} \dfrac{t^3 e^{-st}}{-s} - \dfrac{3t^2 e^{-st}}{s^2} - \dfrac{6t\, e^{-st}}{s^3} - \dfrac{6e^{-st}}{s^4}$

$= -6/s^4$ ☒

---

summary:

- $\mathcal{L}\{1\} = 1/s$
- $\mathcal{L}\{e^{at}\} = 1/(s-a),\ s>a$
- $\mathcal{L}\{t^n\} = n!/s^{n+1}$
- $\mathcal{L}\{\sin at\} = \dfrac{a}{s^2+a^2}$
- $\mathcal{L}\{\cos at\} = \dfrac{s}{s^2+a^2}$

Friday, March 27

thm: linearity of the Laplace transform

1. $\mathcal{L}\{\alpha f(t)\} = \alpha \mathcal{L}\{f(t)\}$
2. $\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha \mathcal{L}\{f(t)\} + \beta \mathcal{L}\{g(t)\}$

proof:

- $\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \int_0^\infty [\alpha f(t) + \beta g(t)] e^{-st}\, dt$

$= \int_0^\infty \alpha f(t) e^{-st}\, dt + \int_0^\infty \beta g(t) e^{-st}\, dt$

$= \alpha \int_0^\infty f(t) e^{-st}\, dt + \beta \int_0^\infty g(t) e^{-st}\, dt$

$= \alpha \mathcal{L}\{f(t)\} + \beta \mathcal{L}\{g(t)\}$ ☒

---

example: Let $f(t) = 4t + 5e^{-7t}$. Find $\mathcal{L}\{f(t)\}$.

- $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}(4t + 5e^{-7t})\, dt$

$= \int_0^\infty 4t e^{-st}\, dt + \int_0^\infty 5e^{-7t} e^{-st}\, dt$

$= 4\int_0^\infty t\, e^{-st}\, dt + 5\int e^{-7t} e^{-st}\, dt$

$= 4 \cdot \mathcal{L}\{t\} + 5\,\mathcal{L}\{e^{-7t}\}$

$= 4 \cdot 1/s + 5 \cdot 1/(s+7)$

$= 4/s + 5/(s+7)$ ☒

---

example: $\mathcal{L}\{\sin(at)\}$, $\mathcal{L}\{\cos(at)\}$

- $e^{ia} = \cos(a) + i\sin(a)$
- $\mathcal{L}\{e^{iat}\} = \mathcal{L}\{\cos(at)\} + i\,\mathcal{L}\{\sin(at)\}$.
- $\mathcal{L}\{e^{iat}\} = \int_0^\infty e^{ait} \cdot e^{-st}\, dt = \int_0^\infty e^{(ai-s)t}\, dt$

$= \dfrac{1}{ai-s} e^{(ai-s)t}\Big]_0^\infty = \dfrac{1}{ai-s}$

- Then, $\dfrac{1}{ai-s} \cdot \dfrac{ai-s}{ai-s} = \dfrac{ai-s}{s^2+a^2} = \dfrac{s}{s^2+a^2} - \dfrac{ai}{s^2+a^2}$

$\Rightarrow \mathcal{L}\{\cos(at)\} = \dfrac{s}{s^2+a^2}$, $\mathcal{L}\{\sin(at)\} = \dfrac{a}{s^2+a^2}$ ☒

---

example: $f(t) = \begin{cases} 1, & 0 \le t < 2 \\ t-2, & t \ge 2 \end{cases}$

- $\mathcal{L}\{f(t)\} = \int_0^2 1 \cdot e^{-st}\, dt + \int_2^\infty (t-2) e^{-st}\, dt$

$= e^{-st}/-s\Big]_0^2 + \int_0^\infty u\, e^{-s(u+2)}\, du$

$= -1/s\,(e^{-2s} - 1) + e^{-2s}\int_0^\infty u \cdot e^{-su}$

$= 1/s\,(1 - e^{-2s}) + e^{-2s} \cdot 1/s^2$

$= 1/s\left(1 - e^{-2s} + \dfrac{e^{-2s}}{2}\right)$

Monday, March 30

first derivative: $\mathcal{L}\{f'(t)\}$

- $\mathcal{L}\{f'(t)\} = \int_0^\infty f'(t) e^{-st}\, dt$

$$\begin{array}{cc} S & I \\ f'(t) & e^{-st} \\ f(t) & -se^{-st} \end{array}$$

$= e^{-st} f(t)\Big]_0^\infty + \int_0^\infty f(t)\, s\, e^{-st}\, dt$

$= [\cancel{e^{-s\infty}} f(\infty) - \cancel{e^{0}} f(0)] + s\int_0^\infty f(t) e^{-st}\, dt$

$= -f(0) + s\,\mathcal{L}\{f(t)\}$ ☒

second derivative: $\mathcal{L}\{f''(t)\}$

- $\mathcal{L}\{f''(t)\} = \int_0^\infty f''(t) e^{-st}\, dt$

$$\begin{array}{cc} S & I \\ f''(t) & e^{-st} \\ f'(t) & -se^{-st} \\ f(t) & s^2 e^{-st} \end{array}$$

$= e^{-st} f'(t) + s f(t) e^{-st}\Big]_0^\infty + s^2\int_0^\infty f(t) e^{-st}\, dt$

$= -[f'(0) + s f(0)] + s^2 \mathcal{L}\{f(t)\}$

$= -f'(0) - s f(0) + s^2 \mathcal{L}\{f(t)\}$ ☒

third derivative: $\mathcal{L}\{f'''(t)\}$

- $\mathcal{L}\{f'''(t)\} = \int_0^\infty f'''(t) e^{-st}\, dt$

$= -f''(0) - s f'(0) - s^2 f(0)t + s^3 \mathcal{L}\{f(t)\}$ ☒ ☒

n^{th} derivative: $\mathcal{L}\{f^{(n)}(t)\}$

- $\mathcal{L}\{f^{(n)}(t)\} = -f^{(n-1)}(0) - s^{n-2} f^{(n-2)}(0) + s^{n-1} f(0) + s^n \mathcal{L}\{f(t)\}$

---

theorem: $\mathcal{L}\{-t f(t)\} = F'(s)$

where $F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t) e^{-st}\, dt$

proof:

- $F(s) = \int_0^\infty f(t) e^{-st}\, dt$

$F'(s) = \dfrac{\partial}{\partial s}\int_0^\infty f(t) e^{-st}\, dt$

$= \int_0^\infty -t \cdot f(t) e^{-st}\, dt$

$= \mathcal{L}\{-t f(t)\}$ ☒

---

theorem: shift theorem (exponential)

- $\mathcal{L}\{e^{at} f(t)\} = F(s-a)$

proof:

- $\mathcal{L}\{e^{at} f(t)\} = \int_0^\infty e^{at} f(t) e^{-st}\, dt = \int f(t) e^{(a-s)t}\, dt$

$= \int_0^\infty f(t) e^{-t(s-a)}\, dt = F(s-a)$ ☒

example: $\mathcal{L}\{e^{at}t^n\}$

- $f(t) = t^n$

$\Rightarrow F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$

$\Rightarrow \mathcal{L}\{e^{at}t^n\} = F(s-a) = \frac{n!}{(s-a)^{n+1}}$ ⊠

Friday, April 3

example: $\mathcal{L}\{e^{at}\sin(bt)\}$

- using the shift theorem:

$$= \frac{b}{(s-a)^2+b^2}$$ ⊠

---

example: $\mathcal{L}\{e^{at}\cos(bt)\}$

- by the shift theorem:

$$= \frac{s-a}{(s-a)^2+b^2}$$ ⊠

---

example: $\mathcal{L}\{t^3+5t-2\}$

$= \mathcal{L}\{t^3\} + 5\,\mathcal{L}\{t\} - 2\,\mathcal{L}\{1\}$

$= \frac{3!}{s^4} + 5\cdot\frac{1}{s^2} - 2\cdot\frac{1}{s}$

$= \frac{6}{s^4} + \frac{5}{s^2} - \frac{2}{s}$ ⊠

---

example: $\mathcal{L}\{e^{2t}(t^3+5t-2)\}$

- $f(t) = t^3+5t-2$

$\Rightarrow \mathcal{L}\{e^{2t}t^3\} + 5\,\mathcal{L}\{e^{2t}t\} - 2\,\mathcal{L}\{e^{2t}\cdot 1\}$

$= \frac{6}{(s-2)^4} + \frac{5}{(s-2)^2} - \frac{2}{(s-2)}$ ⊠

---

example: $\mathcal{L}\{(t^2+4)e^{2t} - e^{-t}\cos(t)\}$

$\Rightarrow \mathcal{L}\{e^{2t}t^2\} + 4\,\mathcal{L}\{e^{2t}\cdot 1\} - \mathcal{L}\{e^{-t}\cos(t)\}$

$= \frac{2}{(s-2)^2} + \frac{4}{(s-2)} - \frac{(s+1)}{(s+1)^2+1}$ ⊠

Recall:

- $\mathcal{L}\{e^{at}\} = \frac{1}{s-a} = F(s)$
- $\mathcal{L}\{-t\,f(t)\} = F'(s) \Rightarrow \mathcal{L}\{t\,f(t)\} = -F'(s)$
- Then,

$\mathcal{L}\{te^{at}\} = -F'(s)$

$= -\left(\frac{1}{s-a}\right)' = \frac{1}{(s-a)^2}$

---

example: $\mathcal{L}\{t\sin(bt)\}$

$= -F'(s)$

- $F(s) = \mathcal{L}\{\sin(bt)\} = \frac{b}{s^2+b^2} = b(s^2+b^2)^{-1}$

$\Rightarrow F'(s) = \frac{-2bs}{(s^2+b^2)^2}$

$\Rightarrow \mathcal{L}\{t\sin(bt)\} = -F'(s) = \frac{2bs}{(s^2+b^2)^2}$ ⊠

---

example: $\mathcal{L}\{t\cos(bt)\}$

$= -F'(s)$

- $F(s) = \mathcal{L}\{\cos(bt)\} = s/s^2+b^2 = s(s^2+b^2)^{-1}$

$\Rightarrow F'(s) = \frac{1}{s^2+b^2} - \frac{-2s^2}{(s^2+b^2)^2} = \frac{s^2+b^2-2s^2}{(s^2+b^2)^2} = \frac{b^2-s^2}{(s^2+b^2)^2}$

$\Rightarrow \mathcal{L}\{t\cos(bt)\} = -F'(s) = \frac{s^2-b^2}{(s^2+b^2)^2}$ ⊠

---

theorem: Laplace inverse

- In general,

$\mathcal{L}^{-1}\{F(s)\} = f(t)$

where $F(s) = \mathcal{L}\{f(t)\}$.

---

example: $\mathcal{L}^{-1}\left\{\frac{3}{s^2+4}\right\}$

$\Rightarrow 3\cdot\mathcal{L}^{-1}\left\{\frac{1}{s^2+2^2}\right\}$

$= \frac{3}{2}\cdot\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\} = \frac{3}{2}\sin(2t)$ ⊠

---

example: $\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+4}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{s}{s^2+2^2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+2^2}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{s}{s^2+2^2}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\}$

$= \cos(2t) + \frac{1}{2}\sin(2t)$ ⊠

---

example: $\mathcal{L}^{-1}\left\{\frac{s+1}{s^2-4}\right\}$

$\Rightarrow \mathcal{L}^{-1}\left\{\frac{s+1}{(s-2)(s+2)}\right\}$

- $\frac{s+1}{(s-2)(s+2)} = \frac{A}{s-2} + \frac{B}{s+2}$

$\Rightarrow s+1 = A(s+2) + B(s-2)$

- $s=-2 \Rightarrow -1 = -4B \Rightarrow B = 1/4$

$s=2 \Rightarrow 3 = 4A \Rightarrow A = 3/4$

$\Rightarrow \mathcal{L}^{-1}\left\{\frac{s+1}{(s-2)(s+2)}\right\} = \mathcal{L}\left\{3/4\cdot\frac{1}{s-2}\right\} + \mathcal{L}\left\{1/4\cdot\frac{1}{s+2}\right\}$

$= 3/4\left\{\frac{1}{s-2}\right\} + 1/4\left\{\frac{1}{s+2}\right\}$

$= 3/4\,e^{2t} + 1/4\,e^{-2t}$ ⊠

§6.2: Solution of initial value problems

example: $y'' - 3y' - 10y = 2$; $y(0) = 1$, $y'(0) = 2$

1. take laplace of both sides:

$\Rightarrow \mathcal{L}\{y'' - 3y' - 10y\} = \mathcal{L}\{2\}$

2. let $\mathcal{L}\{y\} = Y(s)$

$\Rightarrow \mathcal{L}\{y''\} - 3\mathcal{L}\{y'\} - 10\mathcal{L}\{y\} = 2 \cdot \mathcal{L}\{1\}$

$[s^2\mathcal{L}\{y(s)\} - sY(0) - Y'(0)]$

$- 3[s\mathcal{L}\{y(s)\} - Y(0)]$

$- 10Y(s) = 2 \cdot \mathcal{L}\{1\}$

$\Rightarrow s^2Y(s) - s - 2 - 3(sY(s) - 1) - 10Y(s) = 2/s$

$s^2Y(s) - s - 2 - 3sY(s) + 3 - 10Y(s) = 2/s$

3. factor $Y(s)$

$\Rightarrow Y(s)[s^2 - 3s - 10] = 2/s + s - 1$

4. solve for $Y(s)$

$\Rightarrow Y(s) = \dfrac{2/s + s - 1}{s^2 - 3s - 10} = \dfrac{2 + s^2 + s}{s(s^2 - 3s - 10)} = \dfrac{s^2 + s + 2}{s(s-5)(s+2)}$

5. $Y(s) = \mathcal{L}\{y(t)\}$

$\Rightarrow \dfrac{s^2 - s + 2}{s(s-5)(s+2)} = \mathcal{L}\{y(t)\}$

$\Rightarrow y(t) = \mathcal{L}^{-1}\left\{\dfrac{s^2 - s + 2}{s(s-5)(s+2)}\right\}$

- $\dfrac{s^2 - s + 2}{s(s-5)(s+2)} = \dfrac{A}{s} + \dfrac{B}{s-5} + \dfrac{C}{s+2}$

$\Rightarrow s^2 - s + 2 = (s+2)(s-5)A + s(s+2)B + s(s-5)C$

- $s = -2 \Rightarrow 8 = 14C = C = 4/7$
- $s = 5 \Rightarrow 22 = 35B \Rightarrow B = 22/35$
- $s = 0 \Rightarrow 2 = -10A \Rightarrow A = -1/5$

$\Rightarrow y(t) = -1/5\, \mathcal{L}^{-1}\{\frac{1}{s}\} + \frac{22}{35}\mathcal{L}^{-1}\{\frac{1}{s-5}\} + \frac{4}{7}\mathcal{L}\{\frac{1}{s+2}\}$

$= -1/5 + 22/35\, e^{5t} + 4/7\, e^{-2t}$ ☒

---

example: $y'' - 2y' + 2y = e^{-t}$; $y(0) = 0$, $y'(0) = 1$

- $\mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{e^{-t}\}$
- Let $\mathcal{L}\{y(t)\} = Y(s)$

$\Rightarrow s^2\mathcal{L}\{y\} - sy(0) - y'(0) - 2[s\mathcal{L}\{y\} - y(0)] + 2\mathcal{L}\{y\} = \dfrac{1}{s+1}$

$= s^2Y(s) - 1 - 2sY(s) + 2Y(s) = \dfrac{1}{s+1}$

$Y(s)[s^2 - 2s + 2] = \dfrac{1}{s+1} + 1 = \dfrac{s+2}{s+1}$

$\Rightarrow Y(s) = \dfrac{s+2}{\underbrace{(s+1)(s^2 - 2s + 2)}} = \dfrac{s+2}{\underbrace{(s+1)[(s-1)^2 + 1]}_{\text{complete the square}}}$

- $\dfrac{s+2}{(s+1)[(s-1)^2 + 1]} = \dfrac{A}{s+1} + \dfrac{B(s-1) + C}{(s-1)^2 + 1}$

$\Rightarrow s + 2 = A(s-1)^2 + A + B(s-1)(s+1) + C(s+1)$

- $s = 1 \Rightarrow 3 = A + 2C \Rightarrow C = 7/5$

$s = -1 \Rightarrow 1 = 5A \Rightarrow A = 1/5$

$s = 0 \Rightarrow 2 = 2A - B + C \Rightarrow B = -1/5$

$\Rightarrow Y(s) = 1/5\, \mathcal{L}\{1/(s+1)\} + \mathcal{L}\left\{\dfrac{-1/5\, s + 8/5}{(s-1)^2 + 1}\right\}$

$\Rightarrow y(s) = 1/5\, e^{-t} + \mathcal{L}^{-1}\left\{\dfrac{-1/5\, s}{(s-1)^2 + 1^2}\right\} + \dfrac{8}{5}\mathcal{L}^{-1}\left\{\dfrac{1}{(s-1)^2 + 1}\right\}$

$= 1/5\, e^{-t} - 1/5\, \mathcal{L}^{-1}\left\{\dfrac{s - 1 + 1}{(s-1)^2 + 1^2}\right\} + \dfrac{8}{5} e^{t}\sin(t)$

$= 1/5\, e^{-t} - 1/5\left[\mathcal{L}^{-1}\left\{\dfrac{s-1}{(s-1)^2 + 1^2}\right\} + \mathcal{L}^{-1}\left\{\dfrac{1}{(s-1)^2 + 1^2}\right\}\right] + \dfrac{8}{5} e^{t}\sin(t)$

$= 1/5\, e^{-t} - 1/5\left[e^{t}\cos(t) + e^{t}\sin(t)\right] + 8/5\, e^{t}\sin(t)$

$= 1/5\, e^{-t} - 1/5\, e^{t}\cos(t) + 7/5\, e^{t}\sin(t)$ ☒

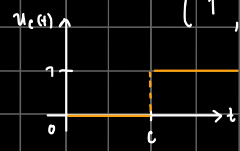
Monday, April 6

Section 6.3: Step Functions

definition: step function

- The step function is defined as

$$u_c(t) = \begin{cases} 0, & 0 \le t < c \\ 1, & t \ge c \quad [c, \infty) \end{cases}$$

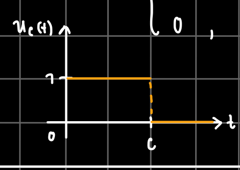


- If we multiply $f(t)$ by $u_c(t)$, then

$$u_c(t)f(t) = \begin{cases} 0, & 0 \le t < c \\ f(t), & t \ge c \quad [c, \infty) \end{cases}$$

- Now, consider

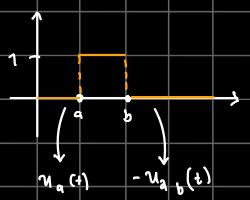
$$1 - u_c(t) = \begin{cases} 1, & 0 \le t < c \\ 0, & t \ge c \end{cases}$$



example: rectangular pulse

- $h(t) = u_a(t) - u_b(t)$
- $u_a(t) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a \end{cases}$

  $u_b(t) = \begin{cases} 0, & 0 \le t < b \\ 1, & t \ge b \end{cases}$



- $u_a(t) - u_b(t) = \begin{cases} 0-0, & 0 \le t < a \\ 1-0, & a \le t < b \\ 1-1, & t \ge b \end{cases}$

  $= \begin{cases} 0, & 0 \le t < a \\ 1, & a \le t < b \\ 0, & t \ge b \end{cases}$

↗ $f(t) = [\text{starting}] + ([\text{next}] - [\text{previous}]) \cdot u_c(t)$

note: we can describe functions in terms of a step function

- $g(t) = \begin{cases} f(t), & a \le t < b \\ 0, & \text{otherwise} \end{cases}$

  $= f(t) \cdot \begin{cases} 1, & a \le t < b \\ 0, & \text{otherwise} \end{cases} = f(t)[u_a(t) - u_b(t)]$

---

example: ↗ $\sin(t)\, u_0(t) + [e^t - \sin(t)]\, u_1(t) + [t^2 - e^t]\, u_5(t)$

- $f(t) = \begin{cases} \sin(t), & 0 \le t < 1 \\ e^t, & 1 \le t < 5 \\ t^2, & t \ge 5 \end{cases}$ ↳ $= \sin(t)[u_0(t) - u_1(t)] + e^t[u_1(t) - u_5(t)] + t^2[u_5(t)]$

  $= \sin(t)[u_0(t) - u_1(t)] + e^t[u_1(t) - u_5(t)] + t^2[u_5(t)]$

  $= \sin(t)[1-1] + e^t[1-1] + t^2 = t^2$

$\Rightarrow \mathcal{L}\{f(t)\} = \mathcal{L}\{\sin(t)[u_0(t) - u_1(t)] + e^t[u_1(t) - u_5(t)] + t^2[u_5(t)]\}$

Wednesday, April 8

generally: for a piecewise function

- $f(t) = \begin{cases} f_0(t), & 0 \le t < t_1 \\ f_1(t), & t_1 \le t < t_2 \\ f_2(t), & t \ge t_2 \end{cases}$
- $f(t) = f_0(t)[\cancelto{1}{u_0(t)} - u_1(t)] + f_1(t)[u_1(t) - u_2(t)] + f_2(t)[u_2(t)]$

  $= f_0(t) + u_1(t)[f_1(t) - f_0(t)] + u_2(t)[f_2(t) - f_1(t)]$
- Now,

  $\mathcal{L}\{u_c(t)\} = \int_0^\infty e^{-st} u_c(t)\, dt = \int_0^c e^{-st} \cancelto{0}{u_c(t)}\, dt + \int_c^\infty e^{-st} \cancelto{1}{u_c(t)}\, dt$

  $= \int_c^\infty e^{-st}\, dt = \left. \frac{e^{-st}}{-s} \right]_c^\infty$

  $= -\frac{1}{s}\left[\cancelto{0}{e^{-\infty}} - e^{-sc}\right] = \frac{e^{-sc}}{s}$ ⊠

---

theorem: shift theorem (step function)

- $\mathcal{L}\{u_c(t) \cdot f(t-c)\} = e^{-sc} \cdot F(s)$

---

example $\mathcal{L}\{u_1(t)(t^2+1)\}$

$= \mathcal{L}\{u_1(t)t^2\} + \mathcal{L}\{u_1(t)\}$

- we need $t^2 \Rightarrow (t-1)^2 = t^2 - 2t + 1$

$\Rightarrow \mathcal{L}\{u_1(t)t^2\} = \mathcal{L}\{u_1(t)[(t-1)^2 + 2t - 1]\}$

$\Rightarrow \mathcal{L}\{f(t)\} = \mathcal{L}\{u_1(t)[(t-1)^2 + 2t - 1]\} + \mathcal{L}\{u_1(t)\}$

$= \mathcal{L}\{u_1(t)(t-1)^2\} + 2\mathcal{L}\{u_1(t) \cdot t\} - \cancel{\mathcal{L}\{u_1(t)\}} + \cancel{\mathcal{L}\{u_1(t)\}}$

$= \mathcal{L}\{u_1(t)(t-1)^2\} + 2\mathcal{L}\{u_1(t)[(t-1)+1]\}$

$= \mathcal{L}\{u_1(t)(t-1)^2\} + 2\mathcal{L}\{u_1(t)(t-1)\} + 2\mathcal{L}\{u_1(t)\}$

$= e^{-s} \cdot \frac{2!}{s^3} + 2 \cdot e^{-s}\frac{1}{s^2} + 2 \cdot \frac{e^{-s}}{s}$

$= \frac{2e^{-s}}{s^3} + \frac{2e^{-s}}{s^2} + \frac{2e^{-s}}{s} = 2e^{-s}\left(\frac{1}{s^3} + \frac{1}{s^2} + \frac{1}{s}\right)$ ⊠

Friday, April 10

example: $f(t) = \begin{cases} \sin(t), & 0 \le t < \pi/2 \\ \cos(t) - 3\sin(t), & \pi/2 \le t < \pi \\ 3\cos(t), & t \ge \pi \end{cases}$. Find $\mathcal{L}\{f(t)\}$

$\Rightarrow f(t) = \sin(t)[\cancelto{1}{u_0(t)} - u_{\pi/2}(t)] + [\cos(t) - 3\sin(t)][u_{\pi/2}(t) - u_\pi(t)] + 3\cos(t)[u_\pi(t)]$

$= \sin(t) + u_{\pi/2}[\cos(t) - 4\sin(t)] + u_\pi[2\cos(t) + 3\sin(t)]$.

$\Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{s^2+1^2} + \mathcal{L}\{u_{\pi/2}[\cos(t) - 4\sin(t)]\} + \mathcal{L}\{u_\pi[2\cos(t) + 3\sin(t)]$

$= \frac{1}{s^2+1^2} + \mathcal{L}\{u_{\pi/2}[\cos(t - \pi/2 + \pi/2) - 4\sin(t - \pi/2 + \pi/2)]\} + \mathcal{L}\{u_\pi[2\cos(t - \pi + \pi) + 3\sin(t - \pi + \pi)]\}$

- Recall $\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$

  $\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y)$

$\Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{s^2+1^2} + \mathcal{L}\{u_{\pi/2}[\cos(t - \pi/2)\cancelto{0}{\cos(\pi/2)} - \sin(t - \pi/2)\cancelto{1}{\sin(\pi/2)}]$
$- 4[\sin(t - \pi/2)\cancelto{0}{\cos \pi/2} + \cancelto{1}{\sin(\pi/2)}\cos(t - \pi/2)]\} +$
$\mathcal{L}\{u_\pi[3[\sin(t-\pi)\cancelto{1}{\cos \pi} + \cos(t-\pi)\cancelto{0}{\sin \pi}] +$
$2[\cos(t-\pi)\cancelto{1}{\cos \pi} - \cancelto{0}{\sin(\pi)}\sin(t-\pi)]]\}$

$= \frac{1}{s^2+1^2} + \mathcal{L}\{u_{\pi/2}[-\sin(t - \pi/2) - 4\cos(t - \pi/2)]\} + \mathcal{L}\{u_\pi[3\sin(t-\pi) + 2\cos(t-\pi)]\}$

$= \frac{1}{s^2+1^2} + e^{-\frac{\pi}{2}s}\left[\frac{-1}{s^2+1^2} - \frac{4s}{s^2+1^2}\right] + e^{-\pi s}\left[\frac{-3}{s^2+1^2} - \frac{2s}{s^2+1^2}\right]$

---

definition: $\mathcal{L}^{-1}\{F(s)e^{-sc}\} = u_c(t) f(t-c)$

---

example: $\mathcal{L}^{-1}\{e^{-2s}/s^2\}$

- $F(s) = 1/s^2$, $c = 2$

$\Rightarrow \mathcal{L}^{-1}\{e^{-2s}/s^2\} = u_2(t)(t-2)$ ⊠

example: $H(s) = 1/s^2 - e^{-s}(1/s^2 + 2/s) + e^{-4s}(4/s^2 + 1/s)$

$\Rightarrow \mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\{1/s^2\} + \mathcal{L}^{-1}\{e^{-s}(1/s^2 + 2/s)\} + \mathcal{L}^{-1}\{e^{-4s}(4/s^2 + 1/s)\}$

$= t - u_1(t)[(t-1) + 2] + u_4(t)[4(t-4) + 1]$

$= t - u_1(t)(t-1) + u_4(t)(4t-15)$ ⊠

---

example: $H(s) = \frac{2s}{s^2+4} - e^{-\frac{\pi}{2}s}\frac{3s+1}{s^2+9} + e^{-\pi s}\frac{s+1}{s^2+6s+10}$

1. $\mathcal{L}^{-1}\{2s/(s^2+4)\} = 2\cos(2t)$
2. $\mathcal{L}^{-1}\{e^{-\frac{\pi}{2}s}\frac{3s+1}{s^2+9}\} = u_{\pi/2}(t)\left[\mathcal{L}^{-1}\left\{\frac{3s}{s^2+9} + 1/(s^2+9)\right\}\right]$

$= u_{\pi/2}(t)[3\cos 3(t-\pi/2) + \sin 3(t-\pi/2)]$

$= u_{\pi/2}(t)[-3\sin(3t) - \cos(3t)]$

3.

Monday, April 13

§6.4: Differential Equations with Discontinuous Forcing Functions

example: $y'' + 2y' + 2y = g(t)$; $g(t) = \begin{cases} 0, & 0 \le t < 1 \\ 2, & t \ge 1 \end{cases}$, $y(0) = 1$, $y'(0) = 0$

- $2g(t) = \begin{cases} 0, & 0 \le t < 1 \\ 1, & t \ge 1 \end{cases} = 2u_1(t)$

$\Rightarrow \mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + 2L(y) = 2\mathcal{L}\{u_1(t)\}$

$= s^2 L(y) - s\cancelto{1}{y(0)} - \cancelto{0}{y'(0)} + 2[sL(y) - \cancelto{1}{y(0)}] + 2L(y) = 2\mathcal{L}\{u_1(t)\}$

$= L(y)[s^2 + 2s + 2] - s + 2 = 2e^{-s}/s$

$\Rightarrow L(y) = \frac{2e^{-s}/s + s + 2}{(s^2 + 2s + 2)}$

$L(y) = \frac{2e^{-s}}{s[(s+1)^2+1]} + \frac{s+2}{(s+1)^2+1}$

$\Rightarrow y(t) = 2\mathcal{L}^{-1}\left\{\underbrace{\frac{e^{-s}}{s[(s+1)^2+1]}}_{*}\right\} + \mathcal{L}^{-1}\left\{\underbrace{\frac{s+2}{(s+1)^2+1}}_{**}\right\}$

- $\frac{1}{s[(s+1)^2+1]} = \frac{A}{s} + \frac{B(s+1)+C}{(s+1)^2+1}$

$\Rightarrow 1 = A[(s+1)^2+1] + [B(s+1)+C]s$

- $s = 0 \Rightarrow 1 = 2A \Rightarrow A = 1/2$
- $s = -1 \Rightarrow 1 = A - C \Rightarrow C = -1/2$
- $s = 1 \Rightarrow 1 = 5A + 2B + C \Rightarrow B = -1/2$

$t \to t-1$

$\Rightarrow \mathcal{L}^{-1}\{*\} = 2\mathcal{L}^{-1}\left\{e^{-s}\left[\frac{1}{2s} - \frac{(s+1)}{2(s+1)^2+1} - \frac{1}{2(s+1)^2+1}\right]\right\}$

$= 2u_1(t)\left[1/2 - 1/2\cos(t-1)e^{-(t-1)} - 1/2\sin(t-1)e^{-(t-1)}\right]$

$= u_1(t)\left[1 - e^{-(t-1)}[\cos(t-1) + \sin(t-1)]\right]$

$\Rightarrow \mathcal{L}^{-1}\{**\} = \mathcal{L}^{-1}\left\{\frac{s+2}{(s+1)^2+1}\right\} = \mathcal{L}^{-1}\left\{\frac{(s+1)-1}{(s-1)^2+1}\right\} = \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}\right\}$

$= e^{-t}[\cos(t) + \sin(t)]$

$\Rightarrow y(t) = e^{-t}[\cos(t) + \sin(t)] + u_1(t)\left[1 - e^{-(t-1)}[\cos(t-1) + \sin(t-1)]\right]$ ⊠

---

useful: trigonometric identities

- $\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$
- $\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y)$

Wednesday, April 15

Chapter 7: Introduction to systems of ODEs

- Recall,

$$ay'' + by' + cy = g(t) \;;\; y(0) = \alpha,\; y'(0) = \beta \qquad (1)$$

- This 2nd order diff eq. can be transformed (reduced) to 2 first order diff eqs.
- Steps: Let ① $x_1 = y$ ② $x_1' = y' = x_2$ $\Big\}$ $y(0) = \alpha \Rightarrow x_1(0) = \alpha$; $y'(0) = \beta \Rightarrow x_1'(0) = \beta$ or $x_2(0) = \beta$
  - solve (1) for $y'' = x_2'$
  - $\Rightarrow x_2' = 1/a\,[\,g(t) - by' - cy\,]$

$\Rightarrow$ Any 2nd order diff. eq can be reduced to two first order diff. eqs.

---

example: Given $y'' + 5y' - 10y = \sin(t)$; $y(0) = 2$, $y'(0) = 4$

Rewrite into a system of 1st order diff eqs.

- Let $x_1 = y \Rightarrow x_1' = y' = x_2 \Rightarrow y'' = x_2'$

1. $x_1' = x_2$
2. $x_2' = \sin(t) + 10y - 5y'$ $\Big\}$ system

- $y(0) = 2 \Rightarrow x_1(0) = 2$
  $y'(0) = 4 \Rightarrow x_2(0) = 4$

- For higher order equations:
  - Let $x_1 = y \rightarrow x_1' = y' = x_2$
    $x_2 = y' \rightarrow x_2' = y'' = x_3$
    $\vdots$
    $x_n = y^{(n-1)} \rightarrow x_n' = y^{(n)} = x_{n+1}$

---

example: Given $\begin{cases} x_1' = 3x_1 - 2x_2 & (1) \quad x_1(0) = 3 \\ x_2' = 2x_1 - 2x_2 & (2) \quad x_2(0) = 1/2 \end{cases}$

Transform this system of 1st order DEs into a 2nd order DE.

(1) solve for x_2 in (1)
- $x_2 = 1/2\,(3x_1 - x_1')$

(2) plug x_2 into (2)
- $x_2' = 2x_1 - (3x_1 - x_1')$
  $x_2' = x_1' - x_1$
- $x_2' = 1/2\,(3x_1' - x_1'')$

$\Rightarrow 1/2\,(3x_1' - x_1'') = x_1' - x_1$

$3x_1' - x_1'' = 2x_1' - 2x_1$

$2x_1 + x_1' - x_1'' = 0$

(3) initial conditions
- $x_1(0) = 3$
- $x_2(0) = 1/2 = 1/2\,[3x_1(0) - x_1'(0)]$

$\Rightarrow 1 = [3 \cdot 3 - x_1'(0)] \Rightarrow 1 = 9 - x_1'(0)$

$\Rightarrow x_1'(0) = 8$

(4) solution: $2x_1 + x_1' - x_1'' = 0$; $x_1(0) = 3$, $x_1'(0) = 8$ ☒

§7.2: Review of Matrices

- Let matrix A and B be two square matrices $(n \times n)$.
- addition: $A + B = a_{ij} + b_{ij}$
- scalar mult. $\alpha A = \alpha \cdot a_{ij}$
- transpose: $A^T = (a_{ij})^T = a_{ji}$
  $(A^T)^T = (a_{ji})^T = a_{ij} = A$
- multiplication: $AB = a_i \cdot b_j$
  $C = c_{ij}$
- We use matrices to represent a system of linear DEs

---

example: $x_1 - x_2 + 3x_3 = 4$
$2x_1 \quad + 5x_3 = 0$
$x_2 - x_3 = 7$

$$\begin{pmatrix} 1 & -1 & 3 \\ 2 & 0 & 5 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 7 \end{pmatrix}$$

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example:

- $x_1' = a(t)x_1 + b(t)x_2 + g_1(t)$
- $x_2' = c(t)x_1 + d(t)x_2 + g_2(t)$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix}$$

$$\vec{x}' = A(t) \cdot \vec{x} + g(t)$$

---

definition: identity matrix (I)

- This a matrix with 1's on the diagonals s.t

$$I = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \vdots \\ 0 & 0 & 1 & \ddots & 0 \\ \vdots & & \cdots & 0 & 1 \end{pmatrix}$$

and $AI = IA = A$.

---

definition: determinant

- $\det(A)$ or $|A|$

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ or } \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$= ad - cb$

definition: inverse

- $\text{inverse}(A) = A^{-1}$
- $AA^{-1} = A^{-1}A = I$

$\Big\}$ when $|A| \neq 0$

---

the following statements are all equivalent:

1. A is invertible
2. $\det(A) \neq 0$
3. A is non-singular
4. row vectors in A are L.I.
5. col. vectors in A are L.I.
6. all eigenvalues of A are nonzero.

---

§ 7.3: Eigenvalues & eigenvectors

definition: eigenvalues & eigenvectors

- For an $n \times n$ matrix (A), λ: scalar, $\vec{v}$: column vector, $\vec{v} \neq \vec{0}$.
- If $AB = \lambda v$, then λ is called an eigenvalue of A and $\vec{v}$ is the corresponding eigenvector.
- Additionally, $\alpha\vec{v}$ is also an eigenvector for $\alpha = 0$.
- To find the eigenvalue(s) and eigenvector(s) of A:

$$A\vec{v} - \lambda\vec{v} = 0 \Rightarrow \vec{v}(A - \lambda I) = 0 \qquad \underbrace{\det(A - \lambda I)}_{\text{characteristic polynomial of degree } n}$$

---

example: Find the eigenvalues and eigenvectors of $A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$.

- $\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 4 = 0$

$$\Rightarrow \lambda^2 - 2\lambda - 3 = 0$$
$$\Rightarrow (\lambda - 3)(\lambda + 1) = 0$$
$$\Rightarrow \lambda_1 = 3,\ \lambda_2 = -1$$

- $\lambda_1 = 3$: $A - \lambda I = \begin{pmatrix} 1-3 & 1 \\ 4 & 1-3 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}$
- $(A - \lambda I)\vec{v}_1 = 0 \Rightarrow \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\Rightarrow -2v_1 + v_2 = 0 \Rightarrow v_2 = 2v_1 \xRightarrow{v_1 = 1} \vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

- $\lambda_2 = -1$: $A - \lambda I = \begin{pmatrix} 1-(-1) & 1 \\ 4 & 1-(-1) \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}$
- $(A - \lambda_2 I)\vec{v}_2 = 0 \Rightarrow \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$2v_1 + v_2 = 0 \Rightarrow v_2 = -2v_1 \xRightarrow{v_1 = 1} \vec{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

---

example: $A = \begin{pmatrix} 2 & -9 \\ 4 & 2 \end{pmatrix}$

- $\begin{vmatrix} 2-\lambda & -9 \\ 4 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 + 36$

$$= \lambda^2 - 4\lambda + 40$$

- $\lambda_{1,2} = \dfrac{4 \pm \sqrt{16 - 160}}{2} = 2 \pm 6i$
- $\lambda_1 = 2 + 6i$: $\begin{pmatrix} 2-(2+6i) & -9 \\ 4 & 2-(2+6i) \end{pmatrix} = \begin{pmatrix} -6i & -9 \\ 4 & -6i \end{pmatrix}$
- $\begin{pmatrix} -6i & 9 \\ 4 & -6i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow -6iv_1 - 9v_2 = 0$

$$\Rightarrow -2iv_1 - 3v_2 = 0 \Rightarrow v_2 = -\frac{2i}{3}v_1$$

  - Let $v_1 = i \Rightarrow v_2 = 2/3 \Rightarrow \vec{v}_1 = \begin{pmatrix} i \\ 2/3 \end{pmatrix}$

- $\lambda_2 = 2 - 6i$: $\begin{pmatrix} 6i & -9 \\ 4 & 6i \end{pmatrix}$

$\Rightarrow 6iv_1 - 9v_2 = 0 \Rightarrow 2iv_1 - 3v_2 = 0 \Rightarrow \vec{v}_2 = \begin{pmatrix} i \\ -2/3 \end{pmatrix}$

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definition: linear independence

- A set of vectors $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k$ are linear independent if $\exists$ a set of real & complex numbers $c_1, c_2, \dots, c_k$ s.t

$$c_1\vec{x}_1 + c_2\vec{x}_2 + \dots + c_k\vec{x}_k = 0.$$

OR

$$\det(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k) = 0$$

---

example: $x_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, x_2 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, x_3 = \begin{pmatrix} -4 \\ 1 \\ -11 \end{pmatrix}$

- Let $A = \begin{pmatrix} 1 & 2 & -4 \\ 2 & 1 & 1 \\ -1 & 3 & -11 \end{pmatrix} \xrightarrow{-2R_1 + R_2} \begin{pmatrix} 1 & 2 & -4 \\ 0 & -3 & 9 \\ -1 & 3 & -11 \end{pmatrix} \xrightarrow{R_1 + R_3} \begin{pmatrix} 1 & 2 & -4 \\ 0 & -3 & 9 \\ 0 & 5 & -15 \end{pmatrix}$

$$\xrightarrow{R_2/3} \begin{pmatrix} 1 & 2 & -4 \\ 0 & 1 & -3 \\ 0 & 5 & -15 \end{pmatrix} \xrightarrow{-5R_2 + R_3} \left(\begin{array}{ccc|c} 1 & 2 & -4 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \text{RREF}$$

$\Rightarrow c_1 + 2c_2 - 4c_3 = 0 \Rightarrow c_1 = 4c_3 - 2c_2 = -2c_3$

$c_2 - 3c_3 = 0 \Rightarrow c_2 = 3c_3$

- If $c_3 = 1$: $c_1 = -2$, $c_2 = 3$
- However, c_1 and c_2 depend on $c_3 \Rightarrow \vec{x}_1, \vec{x}_2$ and $\vec{x}_3$ are L.D. ☒
- Verify $\vec{x}_1, \vec{x}_2$ and $\vec{x}_3$ are L.D.

$$\det\begin{pmatrix} 1 & 2 & -4 \\ 2 & 1 & 1 \\ -1 & 3 & -11 \end{pmatrix} = 1(-11-3) - 2(-22+1) + (-4)(6+1)$$
$$= -14 + 42 - 28$$
$$= 0 \Rightarrow \vec{x}_1, \vec{x}_2, \vec{x}_3 \text{ are LD.} ☒$$

---

example: $\vec{x}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \vec{x}_2 = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}, \vec{x}_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

- $\vec{x}_2 = 2\vec{x}_1 \Rightarrow \vec{x}_1$ & $\vec{x}_2$ are L.D. ☒

§7.4: Basic theory of systems of 1st order linear diff. eqs.

- Let $x_1' = P_{11}(t)x_1 + \dots + P_{1n}(t)x_n + g_1(t)$

  $x_2' = P_{21}(t)x_1 + \dots + P_{2n}(t)x_n + g_2(t)$ (1)

  $\vdots$

  $x_n' = P_{n1}(t)x_1 + \dots + P_{nn}(t)x_n + g_n(t)$

  where $P_{11}(t), \dots, P_{nn}(t)$ are elements of matrix $P(t)$ and $g_1(t), \dots, g_n(t)$ a elements of vector $\vec{g}(t)$.
- (1) can be written in a compact form:

  $$\vec{x}' = P(t)\vec{x} + \vec{g}(t) \qquad (2)$$
- (2) is the general form of a system of 1st order linear diff. eqs.
- If $\vec{g}(t) = \vec{0}$, then (2) is homogeneous:

  $$\vec{x}' = P(t)\vec{x} \qquad (3)$$

---

definition: superposition

- If $\vec{x}_1(t)$ and $\vec{x}_2(t)$ are solutions to the homogeneous system of equations, then any linear combination

  $$c_1\vec{x}_1 + c_2\vec{x}_2$$

  is also a solution.
- $\vec{x}_1(t) = \begin{pmatrix} x_{11}(t) \\ x_{21}(t) \\ \vdots \\ x_{n1}(t) \end{pmatrix}$, $x_2(t) = \begin{pmatrix} x_{12}(t) \\ x_{22}(t) \\ \vdots \\ x_{n2}(t) \end{pmatrix}$

---

example: $\vec{x}_1(t) = \begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix}$, $\vec{x}_2(t) = \begin{pmatrix} e^{-t} \\ -2e^{-t} \end{pmatrix}$. Satisfy $x' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} x$.

- $\vec{x}_1(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}$, $\vec{x}_2(t) = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}$

⇒ $\vec{x}_1'(t) = 3\begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} = \begin{pmatrix} 3 \\ 6 \end{pmatrix} e^{3t} \stackrel{?}{=} \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}$

$= \begin{pmatrix} 3 \\ 6 \end{pmatrix} e^{3t}$ ✓

⇒ $\vec{x}_2'(t) = -\begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{-t} \stackrel{?}{=} \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}$

$= \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{-t}$ ✓

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§7.5: Homogeneous systems of two equation with constant coefficients

- consider $x_1' = ax_1 + bx_2$; $x_1(0) = \tilde{x}_1$

  $x_2' = cx_1 + dx_2$, $x_2(0) = \tilde{x}_2$
- $\vec{x}' = A\vec{x}$; $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\vec{x}(0) = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix}$
- steps to solve IVP:
  1. Find eigenvalues of A: λ_1, λ_2
  2. Find the corresponding eigenvectors $\vec{v}_1, \vec{v}_2$
  3. Form of general solution:

     $\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$
  4. Determine c_1 and c_2 using IC.

---

example: solve $\vec{x}' = A\vec{x}$, $A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$

- eigenvalues: $|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{vmatrix} = 0$

  $= (1-\lambda)^2 - 4 = 0$

  $(1-\lambda-2)(1-\lambda+2) = 0$

  $(-\lambda-1)(-\lambda+3) = 0$

  ⇒ $\lambda_1 = -1$, $\lambda_2 = 3$
- $\lambda_1 = -1$: $\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$ ⇒ $2x_1 + x_2 = 0$

  ⇒ $x_2 = -2x_1$

  ⇒ $\vec{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$
- $\lambda_2 = 3$: $\begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$ ⇒ $-2x_1 + x_2 = 0$

  ⇒ $x_2 = 2x_1$

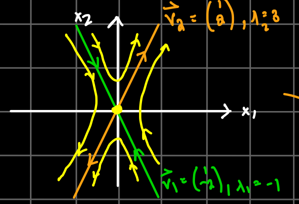
  ⇒ $\vec{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$
- general solution: $\vec{x} = c_1 e^{-t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 e^{3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

  ⇒ $x_1 = c_1 e^{-t} + c_2 e^{3t}$

  ⇒ $x_2 = -2c_1 e^{-t} + 2c_2 e^{3t}$
- as $t \to \infty$:
  - if $c_2 > 0$, then $x_1 \to \infty$

    $x_2 \to \infty$
  - if $c_2 < 0$, then $x_1 \to \infty$

    $x_2 \to -\infty$
  - $\lim_{t\to\infty} \frac{x_1}{x_2} = \frac{1}{2}$
- phase portrait:



system grows

$\lambda > 0$ ⇒ away from $(0,0)$

$\lambda < 0$ ⇒ towards $(0,0)$

system decays

$(0,0)$ is a saddle point because the $\mathrm{sgn}(\lambda_1) \neq \mathrm{sgn}(\lambda_2)$.

---

definition: saddle point

- if A has two real eigenvalues of opposite signs, the origin is called a saddle point.

---

definition: stability

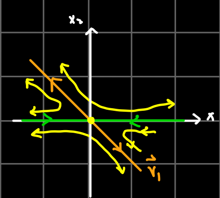
- for solutions nearby the origin or any other point as $t \to \infty$
  1. if the solution diverges from the origin, it is unstable
  2. it the solution tends towards the origin, it is asymptotically stable
  3. it the solution stays nearby but doesn't tend toward the origin, it is stable.

---

example: Let $\lambda_1 = 3$, $\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$; $\lambda_2 = -3$, $\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

- $x_1 = -x_2$

  $-x_1 = x_2$



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example: $x' = \begin{pmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix} x$

- $\begin{vmatrix} 3-\lambda & \sqrt{2} \\ \sqrt{2} & 2-\lambda \end{vmatrix} = (3-\lambda)(2-\lambda) - 2 = 0$

$= 6 + 5\lambda + \lambda^2 - 2 = 0$

$= \lambda^2 + 5\lambda + 4 = 0$

$= (\lambda+1)(\lambda+4) = 0 \Rightarrow \lambda_1 = -1,\ \lambda_2 = -4$

- $\lambda_1 = -1$: $\begin{pmatrix} -2 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$-2c_1 + \sqrt{2}c_2 = 0 \Rightarrow c_2 = \frac{2}{\sqrt{2}} c_1$

$c_1 = \sqrt{2} \Rightarrow c_2 = 2 \Rightarrow \vec{v}_1 = \begin{pmatrix} \sqrt{2} \\ 2 \end{pmatrix}$

- $\lambda_2 = -4$: $\begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

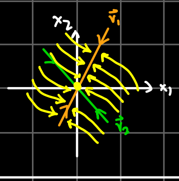
$c_1 + \sqrt{2}c_2 = 0 \Rightarrow c_2 = \frac{-c_1}{\sqrt{2}}$

$c_1 = -\sqrt{2} \Rightarrow c_2 = 1 \Rightarrow \vec{v}_2 = \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix}$

$\Rightarrow x(t) = c_1 e^{-t}\begin{pmatrix} \sqrt{2} \\ 2 \end{pmatrix} + c_2 e^{-4t}\begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix}$ slow $= \vec{v}_1$, fast $= \vec{v}_2$

- $\lim_{t\to\infty} x(t) = 0$ but $c_1 e^{-4t}$ decays faster

use $-\infty$ when $\text{sgn}(\lambda)$ is positive



(0,0) is a sink node

---

trajectory rules:

example: $\lambda_1 = 3,\ \lambda_2 = 4$

1. identify "slow" and "fast" eigenvalues.
   closest to 0 — furthest from 0

   in this case, $|\lambda_1| = 3$ is slower than $|\lambda_2| = 4$.
2. near the origin = tangent to slow eigenvector
3. far from origin = parallel to fast eigenvector.
4. no crossing!

---

summary:

1. if $\lambda_1, \lambda_2 \in \mathbb{R}$ with $\text{sgn}(\lambda_1) \neq \text{sgn}(\lambda_2)$, the origin is a saddle point and unstable.
2. if $\lambda_1, \lambda_2 \in \mathbb{R}$ with $\text{sgn}(\lambda_1) = \text{sgn}(\lambda_2)$, the origin is a node.
   i. if $\lambda_1, \lambda_2 > 0$, this is a source node (unstable). test $\lim_{t\to-\infty} x(t)$
   ii. if $\lambda_1, \lambda_2 < 0$, this is a sink node (stable). test $\lim_{t\to\infty} x(t)$
