SPRING 2025

MATH 231: Linear Algebra

Spaces

Wed, Jan 15

- A central goal in math is abstraction.
- Modern math aims to understand the underlying structures of groups of objects.

Pros:

① more powerful
↳ handle infinite problems
② easier to make connections
③ easily connect areas of math

Cons:

① harder to understand

- Classifying spaces is one way we "abstract" math.
- Most spaces have
  ① objects, A
  ② operations
  ③ axioms

Axioms

(Definition) Axioms are rules that characterize a space.

① commutativity: $a \square b = b \square a$ ← not common

Addition of integers
↓ operation ↓ object

② associativity: $a \square (b \square c) = (a \square b) \square c$

③ identity:
- there exists e in the space where $a \square e = a$ and $e \square a = a$, for any a in the space.
- Examples: $(\mathbb{Z}, +)$ $a + 0 = a$
  $0 + a = a$
  $(\mathbb{N}, \times)$ $5 \times 1 = 5$
  $1 \times 5 = 5$

④ "closed under inverses": → exception: additive identity
- For any a in the space, there is an a^{-1} in the space where.
  $a \square a^{-1} = e$ → identity
  $a^{-1} \square a = e$
- Examples: $(\mathbb{Z}, +)$ $a + (-a) = 0$
  $(\mathbb{R}^{\times}, \times)$ $10 \times \frac{1}{10} = 1$

⑤ distributive property:
- $a \square (b \triangle c) = (a \square b) \triangle (a \square c)$

Practice:

① $\mathbb{R}$ with $+$ and $\times$

What axioms hold?
- commutativity for both
- associativity for both
- identity for $+ : 0$
  $\times : 1$
- closed under inverses for $+$
  $\mathbb{R}^{\times}$ is closed under inverses for $\times$
- distributive property

Note: when all axioms hold we call this a field. ← $(\mathbb{R}, +, \times)$, $(\mathbb{Z}, +, \times)$, $(\mathbb{Z}_p, +_p, \times_p)$

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② $\{1, 2, 3\}$,
$\_ \circ \_ = \max\{\_, \_\}$
$\_ * \_ = \min\{\_, \_\}$

① commutativity ✓
② identities: ✓ max : 1 → $1 \circ 3 = 3$, $1 \circ 2 = 2$
min : 3 → $1 \circ 3 = 1$, $2 \circ 3 = 2$
③ associativity ✓
④ closed under inverses: × → therefore not a field
$a \circ \underline{x} = \max(a, x) = 1$
$a * \underline{x} = \min(a, x) = 3$
⑤ distributive: ✓ $a \circ (b * c) \overset{?}{=} (a \circ b) * (a \circ c)$
- consider 3 possibilities
  1. $a \leq b \leq c$: $a \circ (b * c) = a \circ b = b$. ✓
     $(a \circ b) * (a \circ c) = b * c = b$
  2. $b \leq a \leq c$ ⋮ and so forth
  3. $b \leq c \leq a$

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③ Explain why $(\mathbb{N}, +, \times)$ is not a field.

- No inverses for addition & multiplication.
- No additive identity

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④ $(\mathbb{Z}_7, \times_7, +_7)$ if p wasn't prime, this wouldn't be a field

- $\mathbb{Z}_7 = \{0,1,2,3,4,5,6\} = [6]$
- $a +_7 b = (a+b) \bmod 7$
  - $3 +_7 4 = 7 \bmod 7 = 0$
- $a \times_7 b = (a \times b) \bmod 7$
  - $3 \times_7 4 = 12 \bmod 7 = 5$

Axioms:

- associativity: inherited from $+$ and $\times$.
- identities: $+_7$: 0
  $\times_7$: 1
- inverses: $+_7$: ✓
  given $a \in \mathbb{Z}_7$, $a+(-a)=0$ in $\mathbb{Z}$.
  in $\mathbb{Z}_7$, $-a = 7-a$.

Complex Numbers

(Definition) A complex number is an ordered pair (a,b), $a, b \in \mathbb{R}$, often written as $a+bi$.

- thus,

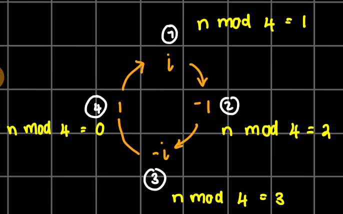
$$\mathbb{C} = \{a+bi ; a, b \in \mathbb{R}\}$$

(Theorem) $\mathbb{C}$ is a vector space over $\mathbb{R}$.

(Checklist)
1. $\mathbb{C}$ satisfies vector addition.
2. $\mathbb{C}$ satisfies scalar multiplication.
   given $a \in \mathbb{R}$, $b, c \in \mathbb{C}$
   $a(b,c) = a(b+ci) = ab + aci = (ab, ac)$
3. The additive identity is $(0,0) = 0+0i = 0$
4. Given $(a,b) \in \mathbb{C}$, the additive inverse is $(-a,-b)$ since $(a-a, b-a) = (0,0)$.
5. 1 is the multiplicative identity from $\mathbb{R}$.
   $1(a,b) = (a,b)$
6. Given $a \in \mathbb{R}$, (b,c) and $(d,f) \in \mathbb{C}$, then
   $a((b,c)+(d,f)) = a(b,c) + a(d,f)$
7. commutativity ✓
8. associativity ✓

Exercises:

1. $x^2 - a^2 = \boxed{(x-a)(x+a)}$ //



2. $x^2 + a^2 = x^2 - a^2(-1)$
   $= x^2 - a^2(i^2)$
   $= x^2 - (ai)^2$
   $= \boxed{(x-ai)(x+ai)}$ //

Vector Spaces

- objects: vectors → ordered list of numbers
  → capture direction & magnitude
- operations: 1. vector addition
  2. scalar multiplication
  3. no vector multiplication

(Vector Space) A vector space over a field $\mathbb{F}$ is a set V with two operations,
1. vector addition,
2. scalar multiplication,

satisfying the following identities (axioms):

Axioms:

1. commutativity:
   - $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ $\forall \vec{u}, \vec{v} \in V$
   - $\alpha \vec{u}(\vec{v}) = \alpha(\vec{u}\vec{v})$ $\forall \vec{u}, \vec{v} \in V$, $\alpha \in \mathbb{F}$
2. associativity:
   - $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$ $\forall \vec{u}, \vec{v}, \vec{w} \in V$
   - $(\alpha\beta)\vec{v} = \alpha(\beta\vec{v})$ $\forall v \in V$, $\forall \alpha, \beta \in \mathbb{F}$
3. additive identity:
   - $\exists \vec{0} \in V$ such that $\vec{v} + \vec{0} = \vec{v}$
4. additive inverses:
   - $\forall \vec{v} \in V$, $\exists \vec{w} \in V$ such that $\vec{v} + \vec{w} = \vec{0}$
5. multiplicative identity:
   - $\exists 1 \in \mathbb{F}$ such that $1\vec{v} = \vec{v}$ $\forall \vec{v} \in V$
6. multiplicative inverses:
   - $\exists a' \in \mathbb{F}$ such that $a' \cdot a = 1$ $\forall a \in \mathbb{F}$.
7. distributive property:
   - $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$ $\forall a \in \mathbb{F}$, $\vec{u}, \vec{v} \in V$ (vector addition)
   - $(a+b)\vec{v} = a\vec{v} + b\vec{v}$ $\forall a, b \in \mathbb{F}$, $\vec{v} \in V$ (field addition)

(Closure) Vector spaces must also satisfy the axioms of closure under vector addition and scalar multiplication.
if $\vec{v}, \vec{w} \in \mathbb{F}$, $\vec{v} + \vec{w} \in \mathbb{F}$
if $\vec{v} \in \mathbb{F}$, $a\vec{v} \in \mathbb{F}$

$V + V \to V$

(Vector Addition) An addition of a set V is a (binary) function that assigns an element $\vec{u} + \vec{v}$ for each pair of $\vec{u}, \vec{v} \in V$.

$\mathbb{F} \cdot V \to V$

(Scalar Multiplication) Scalar multiplication of a set V is a function that assigns $\lambda\vec{v} \in V$ for $\lambda \in \mathbb{F}$ and $v \in V$.

Examples:

1. $\mathbb{F} = \mathbb{Z}_2$, $V = \{(x,y) \mid x,y \in \mathbb{Z}_2\}$

addition : $(x+y) + (u+v) = (x +_2 u, y +_2 v)$

scalar multiplication modulo 2

$V = \{(0,0), (0,1), (1,0), (1,1)\}$

$+_p$	(0,0)	(1,0)	(0,1)	(1,1)
(0,0)	(0,0)	(1,0)	(0,1)	(1,1)
(1,0)	(1,0)	(0,0)	(1,1)	(0,1)
(0,1)	(0,1)	(1,1)	(0,0)	(1,0)
(1,1)	(1,1)	(0,1)	(1,0)	(0,0)

exhaustive proof of
* closure of $+_p$
* additive identity (0,0)
* additive identity
* commutativity

a. associativity: inherited from field (component-wise addition)

b. closure of scalar multiplication:

- $\lambda \in \mathbb{F}$, $x,y \in V$
- $\lambda(x,y) = (\lambda x, \lambda y) \in V$ — essentially inherited from field

since $\lambda x, \lambda y \in \mathbb{F}$. ■

. multiplicative identity:

- Observe that $1(x,y) = (1x, 1y) = (x,y)$ for all $(x,y) \in V$ and $1 \in \mathbb{R}$.

c - Therefore, 1 is the multiplicative identity.

d. distributive property.
i) $\lambda((x,y) + (u,v)) = \lambda(x,y) + \lambda(u,v)$
ii) $(\alpha + \lambda)(x,y) = \alpha(x,y) + \lambda(x,y)$

i. $\forall \lambda \in \mathbb{F}$, $(x,y),(u,v) \in V$

$$\begin{aligned}
\lambda((x,y)+(u,v)) &\overset{\text{vector addition}}{=} \lambda(x+u, y+v) && \text{scalar multiplication} \\
&= (\lambda(x+u), \lambda(y+v)) && \text{distributivity of field.} \\
&= (\lambda x + \lambda u, \lambda y + \lambda v) && \text{reverse vector addition} \\
&= (\lambda x, \lambda y) + (\lambda u + \lambda v) && \text{reverse scalar multiplication} \\
&= \lambda((x,y) + (u,v)) \quad ■
\end{aligned}$$

ii. $\forall \alpha, \lambda \in \mathbb{F}$, $(x,y) \in V$

$$\begin{aligned}
(\alpha+\lambda)(x,y) &\overset{\text{scalar multiplication}}{=} ((\alpha+\lambda)x, (\alpha+\lambda)y) && \text{distributivity} \\
&= (\alpha x + \lambda x, \alpha y + \lambda y) && \text{reverse vector addition} \\
&= (\alpha x, \alpha y) + (\lambda x, \lambda y) && \text{reverse scalar multiplication} \\
&= \alpha(x,y) + \lambda(x,y) \quad ■
\end{aligned}$$

Theorems

(Theorem 1.26) In a vector space V, the additive identity is unique.

Proof :

- Let 0 and 0' be additive identities of V, meaning:
  $v + 0 = v$ and $v + 0' = v$.
- Consider the sum $0 + 0'$:
  $0 + 0' = 0$ (because 0' is an additive identity)
- Similarly, since 0 is an additive identity:
  $0' + 0 = 0'$.
- By transitivity of equality:
  $0' = 0$
- Therefore, 0 and 0' must be equal, proving that the additive identity is unique.

(Corollary) If all elements in a vector space V act as an additive identity, then V has only 1 element and V is called the trivial vector space

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(Theorem 1.27) For any vector in a vector space V, its additive inverse is unique.

Proof :

- Let w and w' be additive inverses of v, meaning:
  $v + w = 0$ and $v + w' = 0$.
- Consider w:
  $w = w + 0$ (by identity property)
- Substitute $v + w'$ for 0:
  $w = w + (v + w')$
- Use associativity:
  $w = (w + v) + w'$
- Since w is an additive inverse:
  $w = 0 + w'$
- By additive identity property:
  $w = w'$.
- Therefore, w and w' must be equal, proving that the additive inverse is unique ■

(Theorem 1.30) Let V be a vector space over a field $\mathbb{F}$, and let $0 \in \mathbb{F}$ (the number 0) be the additive identity of $\mathbb{F}$. Then, for any $v \in V$, $0v$ is the additive identity of V.

Proof: NTS: $v + 0v = v = 0v + v$

- Let $0 \in \mathbb{F}$ and $v \in V$.
- Consider $0v = (0+0)v$
  $= 0v + 0v$ — distribution of scalar multiplication over field addition
- Adding the additive inverse of $0v$:
  $0v + (-0v) = 0v + 0v + (-0v)$ — associativity
  additive inverse: $0 = 0v + (0v + (-0v))$
  $0 = 0v + 0$
- Thus, $0v$ is the additive identity of V. ∎

(Theorem 1.31) Let $\vec{0} \in V$ be the additive inverse and let $a \in \mathbb{F}$. Then $a \cdot \vec{0} = \vec{0}$.

(Theorem 1.32) Let $-1 \in \mathbb{F}$. For $v \in V$, $(-1)v$ (the number -1 times a vector) is the additive inverse of v.

Proof:

- Let $v \in V$.
- Since $-1 \in \mathbb{F}$:
  $-1v + v = (-1+1)v = 0v$.
- By commutativity:
  $v + (-1v) = (1+(-1))v = 0v$.
- Since $1v + 0v = (1+0)v = 1v = v$,
  $0v$ is the additive identity of V.
- Thus, $(-1)v$ is the additive inverse of v.

Subspaces

(Theorem 1.34) A subset is a subspace $U \subseteq V$ of a vector space if and only if U satisfies the following.

additive identity/zero vector → 1. $\vec{0} \in U$

addition closure → 2. $u, v \in U \Rightarrow u+v \in U$

multiplication closure → 3. for $c \in \mathbb{F}$ and $u \in U$, $cu \in U$.

Examples:

1. $P^2 \leq P^3$
2. $\mathbb{R}^2 \leq \mathbb{C}^2$
3. $\{(x,y) \in \mathbb{R}^2 \mid x+y=0\} \leq \mathbb{R}^2$
4. $\{(x,y) \in \mathbb{R}^2 \mid x+y=1\}$ not a v.s.
5. $\mathbb{Z}_2^2 \nleq \mathbb{Z}_3^2$

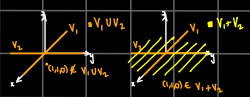
3. $U = \{(x,y) \in \mathbb{R}^2 \mid x+y=0\} \leq \mathbb{R}^2$

(i) Observe that $(0,0) \in U$ since $0+0=0$. ∎

(ii) Let $u = (a,b)$, $v = (x,y) \in U$.
$u + v = (a,b) + (x,y)$
$= (a+x, b+y)$
Observe: $(a+x)+(b+y) = (a+b)+(x+y)$
$= 0+0$ ∎

(iii) Let $\lambda \in \mathbb{R}$ and $v = (x,y) \in U$
$\lambda v = \lambda(x,y) = (\lambda x, \lambda y)$
Observe: $\lambda x + \lambda y = \lambda(x+y) = \lambda 0 = 0$. ∎

Sum of Subspaces



(Definition 1.36) Let $V_1, V_2, \ldots, V_n$ be subspaces of a vector space V. The sum of $V_1, V_2, \ldots, V_n$, denoted $\sum_{i=1}^{\infty} V_i$ is the sum of all subspaces is the sum of all possible elements:

$$V_1 + V_2 + \ldots + V_n = \{v_1 + v_2 + \ldots + v_n \mid v_i \in V_i\}$$

Examples:

1. $P^2, P^3 \subseteq P^4$

$P^2 + P^3 = \{p(x) + q(x) \mid p(x) \in P^2, q(x) \in P^3\}$

Is this just P^3?

$P^3 \subseteq P^2 + P^3$?

Proof:

($\Rightarrow$) Let $p(x) \in P^3$.

- Notice that $0 \in P^2$.
- Since $p(x) + 0 = p(x)$, $p(x) \in P^2 + P^3$

($\Leftarrow$) Let $r(x) = P^2 + P^3$. Then $\exists p(x), q(x)$ in P^2, P^3 such that $r(x) = p(x) + q(x)$.

- Since $p(x) \in P^2$, then $\exists a,b,c \in \mathbb{R}$ s.t.
  $p(x) = a + bx + cx^2$
- Since $q(x) \in P^3$, then $\exists \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \in \mathbb{R}$ s.t.
  $q(x) = \tilde{a} + \tilde{b}x + \tilde{c}x^2 + \tilde{d}x^3$
- Then, $r(x) = p(x) + q(x)$
  $= (a+\tilde{a}) + (b+\tilde{b})x + (c+\tilde{c})x^2 + (0+\tilde{d})x^3$
- Notice that $r(x) \in P^3$ since $(a+\tilde{a}), (b+\tilde{b}) + (c+\tilde{c}) + \tilde{d} \in \mathbb{R}$.

(Theorem 1.40) Suppose $V_1, \dots, V_m$ are subspaces of V. Then $V_1 + \dots + V_m$ is the smallest subspace of V containing $V_1, \dots, V_m$.

Proof:

- Let $\mathcal{U}$ and W be subspaces of a vector space V, and let $S = \mathcal{U} + W = \{u + w \mid u \in \mathcal{U}, w \in W\}$.

1. First, let's prove S is a subspace:
   a. $0 \in S$ since $0 = 0_{\mathcal{U}} + 0_W$ where $0_{\mathcal{U}} \in \mathcal{U}$, $0_W \in W$.
   b. For $s_1, s_2 \in S$: $s_1 = u_1 + w_1$ and $s_2 = u_2 + w_2$ where $u_1, u_2 \in \mathcal{U}$, $w_1, w_2 \in W$
   $s_1 + s_2 = (u_1 + w_1) + (u_2 + w_2)$
   $= (u_1 + u_2) + (w_1 + w_2) \in S$ since $u_1 + u_2 \in \mathcal{U}$, $w_1 + w_2 \in W$.
   c. For $c \in \mathbb{F}$ and $s = u + w \in S$: $cs = c(u+w) = cu + cw \in S$
   since $cu \in \mathcal{U}$, $cw \in W$.
2. Next, let's show $\mathcal{U}, W \in S$.
   - For any $u \in \mathcal{U}$: $u = u + 0_W \in S$.
   - For any $w \in W$: $w = w + 0_{\mathcal{U}} \in S$.
3. Finally, let's show S is the smallest such subspace.
   - Let T be any subspace containing both $\mathcal{U}$ and W.
   - For any $s \in S$: $s = u + w$ where $u \in \mathcal{U} \subseteq T$ and $w \in W \subseteq T$.
   - Since T is a subspace, it's closed under addition, so $u + w \in T$.
   - Therefore, $S \subseteq T$.

- We've shown that S is a subspace containing $\mathcal{U}$ and W, and any other subspace containing $\mathcal{U}$ and W must contain S.
- Therefore, S is the smallest subset containing both $\mathcal{U}$ and W.

Examples:

1. $V = \mathbb{R}^3$
   $V_1 = \{(0, y, z) \mid y + z = 0,\ y, z \in \mathbb{R}\}$
   $V_2 = \{(x, 0, 0) \mid x \in \mathbb{R}\}$
   $V_3 = \{(0, 2y, 0) \mid y \in \mathbb{R}\}$
   $V_1 + V_2 + V_3 = \{v_1 + v_2 + v_3 \mid v_i \in V_i\}$
   $(1,1,1) \in V_1 + V_2 + V_3$
   $(0,-1,1) + (1,0,0) + (0,2,0)$

Direct Sums $\oplus$ Mon, Feb 10

(Definition) Suppose $V_1, \dots, V_m$ are subspaces of V.
- The sum $V_1 + \dots + V_m$ is called the direct sum if each element can be written in only one way.
- If $V_1 + \dots + V_m$ is a direct sum, we write
  $V_1 \oplus \dots \oplus V_m$.

Examples:

1. - $V = \mathbb{R}^2$
   - $V_1 = \{(x, 0) \mid x \in \mathbb{R}\}$
   - $V_2 = \{(0, y) \mid y \in \mathbb{R}\}$
   - $V_1 + V_2 = V$
   - Take $(u, w) \in V$.
     $(u, w) = (u, 0) + (0, w)$ — the only way we can decompose V into V_1, V_2.
   - So, $V_1 \oplus V_2 = V$

2. - $V = \mathbb{R}^2$
   - $V_1 = \{(x, x) \mid x \in \mathbb{R}\}$
   - $V_2 = \{(0, y) \mid y \in \mathbb{R}\}$
   - Take $(u, w) \in V$.
     $(u, w) = (u, u) + (0, w - u)$
   - So, $V_1 \oplus V_2 = V$

minimize redundancy $\dim(V_1) + \dim(V_2) = \dim(V)$

(Theorem 1.45) - Suppose $V_1, \dots, V_m$ are subspaces of V.
- Then $V_1 + \dots + V_m$ is a direct sum if and only if the only way to write 0 as a sum of $V_1 + \dots + V_m$ is by taking $v_i = 0$ for all i.

Geometrically: split the space evenly with no overlap (except 0)

$v_1 + v_2 = 0 \iff v_1 = v_2 = 0$.

Proof:

$\boxed{\Rightarrow}$
- If $V_1 + \dots + V_m$ is a direct sum, then all vectors have a unique representation, by definition.
- $0 \in V$.
- For each i, $0 \in V_i$ (thm 1.34). Notice that $\underbrace{0 + \dots + 0}_{m \text{ times}} = 0$
- Since $V_1 + \dots + V_m$ is a direct sum, by definition, this must be the only way to write it.

$\boxed{\Leftarrow}$
- Suppose there is only one way to represent $0 \in V_1 + \dots + V_m$, which is $\underbrace{0 + \dots + 0}_{m \text{ times}} = 0$.
- But suppose, for contradiction, $V_1 + \dots + V_m$ is NOT a direct sum.
- Then $\exists v \in V_1 + \dots + V_m$ s.t.
  $v = v_1 + \dots + v_m$ AND $v = w_1 + \dots + w_m$ where $v_i \neq w_i$ for some i.
  (since this isn't a direct sum, v has two different representations)
- Then, $-v = -v_1 + \dots + -v_m$ by thm 1.32.
- As a result, $v + (-v) = (w_1 + (-v_1)) + \dots + (w_m + (-v_m))$
  $0 = 0 + \dots + 0$
- By then, $w_i + (-v_i) = 0 \Rightarrow w_i = v_i$ for all i.

Example:

$V = \mathbb{R}^2$

$V_1 = \{(x, x) \mid x \in \mathbb{R}\}$ → line $y = -x$

$V_2 = \{(x, -x) \mid x \in \mathbb{R}\}$ → line $y = x$

$y = -x \perp y = x$

Theorem: $V = V_1 \oplus V_2$

Proof:

- To prove $V = V_1 \oplus V_2$, by theorem 1.45, we need to show that if $v_1 + v_2 = 0$ where $v_1 \in V_1$ and $v_2 \in V_2$, then $v_1 = v_2 = 0$.
- Let $v_1 = (a, a) \in V_1$ and $v_2 = (b, -b) \in V_2$.
- If $v_1 + v_2 = 0$, then:

$$(a, a) + (b, -b) = (0, 0)$$
$$(a+b, a-b) = (0, 0)$$

- This means:

$$a + b = 0 \quad (1)$$
$$a - b = 0 \quad (2)$$

- Adding equation (1) and (2):

$$2a = 0$$
$$a = 0$$

- Substituting into (2):

$$b = 0.$$

- Therefore, $v_1 = (0, 0)$ and $v_2 = (0, 0)$.
- By theorem 1.45, $V = V_1 \oplus V_2$.

(Theorem 1.46) - Suppose U and W are subspaces of V.
- Then $U + W$ is a direct sum $\iff U \cap W = \{0\}$

Proof:

$\boxed{\Rightarrow}$ - Suppose $\exists v \in U \cap W$ where $v \neq 0$.
- Then, $-v \in U \cap W$ since both are vector spaces.
- So $0 = \underset{\in U}{v} + \underset{\in W}{(-v)}$ and by thm. 1.45 $U + W$ is not a direct sum.

$\boxed{\Leftarrow}$ -

2A. Span and Linear Independence

Wed, Feb 12

(Definition) A linear combination of a list of vectors $v_1, v_2, \ldots, v_m$ in a space V is a vector of the form.

$$\underbrace{a_1 v_1 + a_2 v_2 + \ldots + a_m v_m}_{\text{sum of scaled vectors}} \text{ for } a_i \in \mathbb{F}$$

(Definition) A span is the set of all linear combinations of a list of vectors $v_1, v_2, \ldots, v_m$ in V.

$$\text{span}(v_1, \ldots, v_m) = \{a_1 v_1 + \ldots + a_m v_m \mid a_i \in \mathbb{F}, v_i \in V\}$$

Examples:

1. $\text{span}\{(1,0,0)\}$, $(1,0,0) \in \mathbb{R}^3$

$\text{span}\{(1,0,0)\} = \{(x, 0, 0) \mid x \in \mathbb{R}\}$

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2. $\text{span}\{(1,0,0)\}$, $(1,0,0) \in \mathbb{F}_3^2$

$\text{span}\{(1,0,0)\} = \{(0,0,0), (1,0,0), (2,0,0)\}$

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3. $\text{span}\{(1,2,0), (0,-1,1)\}$ where there are elements in $\mathbb{R}^3$.

$$a(1,2,0) + b(0,-1,1)$$
$$= (a, 2a, 0) + (0, -b, b)$$
$$= (a, 2a-b, b)$$

---

see HW 9 #1

4. Find the basis vectors of vector space

$$V = \{(x,y,z) \mid x+y+z = 0,\ x,y,z \in \mathbb{R}\}$$

1. Understand and express the conditions.
   - $x + y + z = 0$
2. Solve for the free variables.
   - $x = -y - z$
   - $(x, y, z) \rightarrow (-y-z, y, z)$
3. Write the general element as a linear combination.
   - $(-y-z, y, z) = (-y, y, 0) + (-z, 0, z)$
     $= y(-1, 1, 0) + z(-1, 0, 1)$
4. Identify the basis vectors.
   - $V = \text{span}\{(-1,1,0), (-1,0,1)\}$

$W = \text{span}\{(-1,1,0,0), (0,0,1,1)\}$

5. $W = \{(a,b,c,d) \in \mathbb{R}^4, a+b=0, c-d=0)$

$a + b = 0 \Rightarrow a = -b$; $c - d = 0 \Rightarrow c = d$

$(a,b,c,d) \rightarrow (-b,b,d,d) = (-b,b,0,0) + (0,0,d,d)$

$b(-1,1,0,0)$
$d(0,0,1,1)$

(Theorem 2.6) The span is the smallest subspace containing the list of vectors.

Proof:

- Let $v_1, \dots, v_m$ be a list of vectors in V.
- We aim to show that $\text{span}(v_1, \dots, v_m)$ is the smallest subspace containing the list.

1. Observe that $0 \in \text{span}(v_1, \dots, v_m)$ since

$$0 = 0v_1 + \dots + 0v_m.$$

2. Suppose $u, w \in \text{span}(v_1, \dots, v_m)$. Then $\exists a_i, b_i \in \mathbb{F}$ s.t

$$u = a_1v_1 + \dots a_mv_m \text{ and } w = b_1v_1 + \dots + b_mv_m.$$

- Then $u + w = (a_1 + b_1)v_1 + \dots + (a_m + b_m)v_m$.
- Since $a_i + b_i \in \mathbb{F}$ $\forall i$, $u + w \in \text{span}(v_1, \dots, v_m)$ □
- Now suppose that $u \in \text{span}(v_1, \dots, v_m)$ and $\lambda \in \mathbb{F}$. Then $\exists a_i \in \mathbb{F}$ s.t

$$u = a_1v_1 + \dots + a_mv_m.$$

3. Observe that $\lambda u = \lambda(a_1v_1 + \dots + a_mv_m) = a_1\lambda v_1 + \dots + a_m\lambda v_m$.

- Since $\mathbb{F}$ is a field, $a_i\lambda \in \mathbb{F}$ $\forall i$.
- Thus, $\lambda u \in \text{span}(v_1, \dots, v_m)$. □
- Hence, $\text{span}(v_1, \dots, v_m)$ is a subspace.

proof by containment

- Now, let's show that it's the smallest such subspace.

1. Observe that each v_k in our list is present in the $\text{span}(v_1, \dots, v_m)$ since $v_k = 0v_1 + 0v_2 + \dots v_k + \dots + 0v_m$.
2. Every subspace that contains $\{v_1, \dots, v_m\}$ must contain the $\text{span}(v_1, \dots, v_m)$ because it must contain all linear combinations due to closure of addition and scalar multiplication.

- Thus, $\text{span}(v_1, \dots, v_m)$ is the smallest subspace containing the list. □

(Corollary 2.6*) Any subspace containing $\{v_1, \dots, v_m\}$ must contain $\text{span}(v_1, \dots, v_m)$.

(Definition) If $\text{span}(v_1, \dots, v_m) = V$ then we say "$v_1, \dots, v_m$ span V".

(Definition) A vector space is finite dimensional if is has a finite spanning set

---

(Definition) A vector space is infinite dimensional if it's not finite dimensional.

(Definition) A list $v_1, \dots, v_m$ in V is linearly independent if

$$a_1v_1 + \dots + a_mv_m = 0 \Rightarrow a_1 = \dots = a_m = 0.$$

Note: The empty list is considered linearly independent.

if the only solution to $av_1 + \dots + a_mv_m = 0$ is $a_1 = \dots = a_m = 0$

(Definition) A list of vectors is linearly dependent if it is not linearly independent.

find a non-trivial vector s.t $a_1v_1 + \dots + a_mv_m = 0$.

Examples:

1. $V = \mathbb{R}^3$, $\{(0,1,1), (1,1,0), (1,0,1)\}$

Linearly independent?

- $a(0,1,1) + b(1,1,0) + c(1,0,1) = (0,0,0)$

$$(0,a,a) + (b,b,0) + (c,0,c) = (0,0,0)$$

$$(b+c, a+b, a+c) = (0,0,0)$$

- So, $b + c = 0 \Rightarrow c = -b$

$a + b = 0 \Rightarrow a = -b = c$

$a + c = 0 \Rightarrow a = -c$

- $a = c$ and $a = -c \Rightarrow a = -a$ $\therefore a + a = 0 \Rightarrow a = 0$.
- Since the field is $\mathbb{R}$.
- If $a = 0$, then $b = 0$ and $c = 0$.
- Thus, they are linearly independent

---

you can remove redundant vectors without changing the span

↳ reduce lin. dep set → lin. ind.

(Lemma 2.19) Linear Dependence Lemma

- Suppose $v_1, \dots, v_m$ is a linearly dependent list in V, then $\exists k \in \{1, 2, \dots, m\}$ s.t

finite dimensional

$$v_k \in \text{span}(v_1, \dots, v_{k-1}).$$

- Furthermore, if k satisfies the condition above and the k^{th} term is removed, then the span of the remaining list is the same.

$$\text{span}(v_1, \dots, v_m) = \text{span}(v_1, \dots, v_{k-1}, v_{k+1}, \dots, v_m)$$

Examples:

1. $V = \mathbb{R}^3$ $\{\underset{v_1}{(1,1,1)}, \underset{v_2}{(2,2,2)}, \underset{v_3}{(9,9,0)}\}$

- $(2,2,2) \in \text{span}(1,1,1)$

---

2. $V = \mathbb{R}^3$ $\{\underset{v_1}{(1,0,0)}, \underset{v_2}{(0,1,0)}, \underset{v_3}{(1,1,0)}\}$

- $(1,1,0) \in \text{span}\{(1,0,0), (0,1,0)\}$

Lemma 2.19 Proof: Constructive

- Because $v_1, \dots, v_m$ is lin. dep., $\exists a_1, \dots, a_m \in \mathbb{F}$, not all zero s.t.

$$a_1v_1 + \dots + a_mv_m = 0.$$

- Let k be the largest element in $\{1, \dots, m\}$ such that $a_k \neq 0$.

$\Rightarrow a_1v_1 + \dots a_kv_k = 0$

$$v_k = \frac{a_1v_1 + \dots + a_{k-1}v_{k-1}}{-a_k} = \frac{1}{-a_k}(a_1v_1 + \dots + a_{k-1}v_{k-1}) \quad \square$$

Lemma 2.19 Proof: Contrapositive − No $k \Rightarrow$ linearly independent

- Suppose $\nexists k$ s.t. $v_k \in \operatorname{span}(v_1, \dots, v_{k-1})$ for a list of $\{v_1, \dots, v_m\}$.
- Now consider a set $\{a_i\} \in \mathbb{F}$ s.t.

$$a_1v_1 + \dots + a_mv_m = 0.$$

- If all $a_i = 0$, we're done.
- Thus, suppose $\exists a_i \neq 0$.
- Let k be the largest element in $\{1, \dots, m\}$ such that $a_k \neq 0$.

$\Rightarrow a_1v_1 + \dots a_kv_k = 0$

$$v_k = \frac{a_1v_1 + \dots + a_{k-1}v_{k-1}}{-a_k} = \frac{1}{-a_k}(a_1v_1 + \dots + a_{k-1}v_{k-1}) \quad \square$$

Replacement theorem

fundamental ingrediants ⊆ complete set of ingrediants
↓ can make any recipe

(Theorem 2.22) length of linearly independent list ≤ length of spanning list

lemma 2.19 proved we can shrink a spanning set into a linearly independent set.

- G - generates/spans V, $\operatorname{span}(G) = V$, $|G| = n$ − spanning list could be linearly dependent.
- L - linearly independent, $|L| = m$ − linearly independent list might not span

1) $m \leq n$

2) $\exists H \subseteq G$, s.t. $L \cup H$ spans V.

$\operatorname{span}(L \cup H) = V$, $|H| = n - m$

Proof:

- Suppose $G = \{w_1, \dots, w_n\}$ spans V.

Suppose $L = \{v_1, \dots, v_m\}$ is a linearly independent list in V.

We want to prove that $m \leq n$.

2. Add u_1 to get $\{u_1, w_1, \dots, w_n\}$
   - This list is linearly dependent (since G spans V and $u_1 \in V$)
   - Some vector is a linear combination of the previous ones.
   - Note that it's not u (it's first and $u_1 \neq 0 \rightarrow$ linearly independent)
   - Thus, it's one of the w's. By the linear dependence lemma, we can remove it while maintaining span.
3. For each u_k ($k = 2$ to m):
   - Add u_k after $u_1, \dots, u_{k-1}$ such that we have $\{u_1, \dots, u_{k-1}, u_k, w_1, \dots, w_n\}$
   - Note that we can't remove any u_i for $i \leq k$ (they're linearly independent)
   - So, remove another w, while maintaining span.
4. After m steps:
   - Added m u's.
   - Removed m w's.
   - To have enough w's to remove, we would need $n \geq m$
- Therefore, $m \leq n$. ■

(Theorem 2.25) Every subspace of a finite dimensional vector space is finite-dimensional (it has a finite spanning set).

Proof:

- Suppose V is finite dimensional and $\mathcal{U}$ is a subspace.
- $\mathcal{U}$ has a linearly independent spanning list. (basis)
- Since $\mathcal{U} \subseteq V$, that list in $\mathcal{U}$ is linearly independent in V
- By the replacement theorem, its size is smaller than the spanning set of V.

Section 2B:

we can create new basis elements by taking linear combinations of existing ones, maintaining LI.

(Definition) A basis is a linearly independent (non-redundant) spanning (sufficient) set of a space.

---

criterion for basis

(Theorem 2.28) A list $v_1, \dots, v_n$ of vectors in V is a basis, if and only if every $v \in V$ can be written in the form

2.29 $$v = a_1v_1 + \dots + a_nv_n \rightarrow \text{spanning}$$

for a unique choice of $\{a_1, \dots, a_n\} \subset \mathbb{F}$.

↓ linear independence $\vec{0} = 0v_1 + \dots + 0v_m$

Proof:

$\Rightarrow$ If $v_1, \dots, v_m$ is a basis, then every $v \in V$ has a unique representation.

- Let $v_1, \dots, v_n$ be a basis of V.
- Let $v \in V$.
- Since the basis spans V, then $\exists \{a_1, \dots, a_n\} \subset \mathbb{F}$ s.t. (by definition)

$$v = a_1v_1 + \dots + a_nv_n \quad (1)$$

- Since the vectors are linearly independent, suppose that

$$a_1v_1 + \dots + a_nv_n = b_1v_1 + \dots b_nv_n$$

- Subtracting the two equations gives

$$(a_1 - b_1)v_1 + \dots + (a_n - b_n) = 0.$$

- By linear independence, all the coefficients must be zero:

$$a_1 - b_1 = 0, \dots, a_n - b_n = 0$$

- Hence $a_i = b_i$ for all i, proving the uniqueness of the representation.

$\Leftarrow$ If every $v \in V$ can be written uniquely as $v = a_1v_1 + \dots + a_nv_n$, then $v_1, \dots, v_n$ is a basis.

- Let $\{v_1, \dots, v_n\} \in V$ s.t. for any $v \in V$, we have

$$v = a_1v_1 + \dots + a_nv_n$$

for a unique choice of $\{a_1, \dots, a_n\}$.

- The hypothesis states that every vector $v \in V$ can be expressed in the form $v = a_1v_1 + \dots + a_nv_n$. By definition, the list $v_1, \dots, v_n$ spans V.
- Assume, for the sake of contradiction, that the list is not linearly independent. Then, there exists a nontrivial relation

$$a_1v_1 + \dots + a_nv_n = 0$$

where not all a_i are zero.

- Notice that the zero vector can be trivially represented as:

$$0 = 0v_1 + \dots + 0v_n.$$

- By the hypothesis of unique representation, these two representations must be identical.
- Therefore, we have $a_1 = 0, \dots, a_n = 0$, contradicting the assumption that not all a_i are zero. Thus, the list must be linearly independent.
- Since the list $v_1, \dots, v_n$ spans V and is linearly independent, it is a basis for V.

every spanning list contains a basis

(Theorem 2.30) Every spanning list in a vector space can be reduced to a basis of a vector space.

Proof:

- Let V be a vector space.
- Let $S = \{v_1, \dots, v_n\}$ be a spanning set.
  1. If s is linearly independent, it's a basis and we're done.
  2. So assume it's linearly dependent.

(Theorem 2.31) Every finite dimensional vector space has a basis.

Proof:

- Let V be a finite dimensional vector space.
- Observe that V is a finite dimensional subspace of V since $V \subseteq V$.
- Then, by theorem 2.25, it has a finite spanning set.
- By theorem 2.30, it has a finite basis. □

(Theorem 2.32) Every linearly independent list of vectors in a finite dimensional vector space can be extended to a basis.

Proof:

- Let $\mathcal{U} = \{u_1, \dots, u_k\}$ be a linearly independent list in a vector space V.
- If $\text{span}(\mathcal{U}) \neq V$, then $\exists\, v \in V$ s.t. $v \notin \text{span}(\mathcal{U})$.
- Define $\mathcal{U}_1 = \mathcal{U} \cup \{v\}$
  - If $\text{span}(\mathcal{U}_1) = V$, we're done.
  - If not, repeat by finding $v \notin \text{span}\{\mathcal{U}\}$.
  - This will terminate because V is finite dimensional
  - By theorem 2.31, V has a basis of length n. This is a spanning set.
  - By theorem 2.22, this process terminates in at most $n-k$ steps because it's linearly independent and at least $n-k$ steps because it's a spanning list.
  - Hence, it terminates in $n-k$ steps and produces a set

$$\{u_1, \dots, u_k, v_1, \dots, v_{n-k}\}$$

which is linearly independent and spanning.

---

so any linearly independent list of basis length is itself a basis.

(Corollary) All basis of a finite dimensional vector space V are the same length.

Proof:

- A finite dimensional vector space has a basis of, say, length n.
- Any other basis is a spanning set and lin. independent.
- By thm. 2.22, it has to be length n. □

(Theorem 2.33) Suppose V is finite-dimensional and $\mathcal{U}$ is a subspace of V. Then, there exists a subspace W of V s.t. $V = \mathcal{U} \oplus W$.

Proof:

- Let V be a finite-dimensional vector space.
- Let $\mathcal{U} \subseteq V$. Thus, $\mathcal{U}$ has a basis

$$B_u = \{u_1, \dots, u_k\}.$$

- Observe that B is linearly independent in V.
- By theorem 2.32, B can be extended to a basis of V s.t.

$$B_V = \{u_1, \dots, u_k, w_1, \dots w_{n-k}\}$$

- Define $W = \text{span}(\{w_1, \dots, w_{n-k}\})$
- Now, let's show $\mathcal{U} \oplus W = V$.
- First, show $\mathcal{U} \cap W = \{0\}$:
  - Suppose $v \in \mathcal{U} \cap W$. Then $\exists\ \{a_1, \dots, a_k\}$ and $\{b_1, \dots, b_k\}$ s.t.

$$v = a_1 u_1 + \dots + a_k u_k$$
$$v = b_1 w_1 + \dots + b_{n-k} w_{n-k}$$

  - Subtracting:

$$0 = (a_1 - b_1)(u_1 - w_1) \dots (a_k - b_{n-k})(u_k - w_{n-k})$$

  - Since B_V forms a basis of V, we must conclude $a_i = b_i = 0\ \forall i$.
  - Thus, $v = 0$
- Now, show that $\forall v \in V$, $v = u + w$, $u \in \mathcal{U}$, $w \in W$.
  - Suppose $v \in V$.
  - Since B_V is a basis of V, $\exists\, a_i$ such that

$$v = \underbrace{a_1 u_1 + \dots + a_k u_k}_{u \in \mathcal{U}} + \underbrace{a_{k+1} w_1 + a_n w_{n-k}}_{w \in W}$$

  - Define $u = a_1 u_1 + \dots + a_k u_k \in \mathcal{U}$.

$$w = a_{k+1} w_1 + \dots + a_n w_{n-k} \in W.$$

- So $\mathcal{U} \oplus W = V$. □

(Theorem 2.34) Any two bases of a finite-dimensional vector space have the same length.

Proof:

- Let V be a finite dimensional vector space.
- Let B_1, B_2 be two bases of V.
- Then, B_1 is a linearly independent list and B_2 is a spanning list.
- By theorem 2.22, $|B_1| \leq |B_2|$.
- Because B_1 is spanning and B_2 is linearly independent, $|B_1| \geq |B_2|$.
- Thus $|B_1| = |B_2|$. □

Dimension:

(Definition) The dimension of a finite dimensional vector space is the length of any basis.
The dimension of a finite dimensional vector space is denoted as $\dim(V)$.

(Theorem 2.37) If V is a finite dimensional vector space and $\mathcal{U}$ is a subspace, then

$$\dim(\mathcal{U}) \leq \dim(V)$$

Proof:

- Let V be a finite dimensional vector space.
- Let $\mathcal{U}$ be a subspace of V.
- Let B be a basis of $\mathcal{U}$.
- This is a linearly independent list in V.
- By theorem 2.32, B can be extended to a basis in V.
- By definition of dimension, it follows that $\dim(\mathcal{U}) \leq \dim(V)$. □

(Theorem 2.38) Suppose V is a finite dimensional vector space. Then every linearly independent set of vectors of length $\dim(V)$ is a basis.

(Proof)

- Suppose $\dim(V) = n$ and let $\{v_1, \dots v_n\}$ be a lin. ind. list in V.
- This list can extend to a basis by theorem 2.32.
- By definition of dimension, it must be length n, so the extension adds no elements. □

(Theorem 2.39) Given a subspace $\mathcal{U}$ of finite dimensional vector space with $\dim(\mathcal{U}) = \dim(V)$, then $\mathcal{U} = V$.

(Theorem 2.42) Suppose V is finite dimensional. Every spanning list of length $\dim(V)$ is a basis.

Proof:

- Suppose S is a spanning set of length $\dim(V)$.
- A spanning set can be reduced to a basis.
- Since all bases are length $\dim(V)$, reducing S to a basis cannot remove any elements. □

(Theorem 2.43) If V_1 and V_2 are subspaces of a finite dimensional vector space V, then

$$\dim(V_1+V_2) = \dim(V_1) + \dim(V_2) - \dim(V_1 \cap V_2)$$

Proof:

- Let $\{v_1, \dots, v_m\}$ be a basis of $V_1 \cap V_2$.
- This is linearly independent in V_1 and so it can be extended to a basis in V_1. Similarly, this is true in V_2.
- $V_1 : \{v_1, \dots, v_m, u_1, \dots, u_k\}$
- $V_2 : \{v_1, \dots, v_m, w_1, \dots, w_\ell\}$
- $\dim(V_1+V_2) = k+\ell-m$
- For any $v \in V_1+V_2$:

$$v = (a_1v_1 + \dots + a_mv_m + a_{m+1}u_1 + \dots a_{m+k}u_k) + (b_1v_1 + \dots + b_mv_m + b_{m+1}w_1 + \dots b_{m+\ell}w_\ell)$$
$$= (a_1+b_1)v_1 + \dots (a_m+b_m)v_m + a_{m+1}u_1 + a_{m+k}u_k + b_{m+1}w_1 + \dots b_{m+\ell}w_\ell$$
$$\Rightarrow \dim(V_1+V_2) = m+k+\ell$$

- $\dim(V_1) = m+k$
- $\dim(V_2) = m+\ell$
- $\dim(V_1 \cap V_2) = m$
- Therefore,

$$\dim(V_1+V_2) = \dim(V_1) + \dim(V_2) - \dim(V_1 \cap V_2) \quad \square$$

Chapter 3: Linear Maps

Visualization:
- stretch, rotate, skew
- grid lines remain straight & parallel
- origin remains fixed

(Definition) A linear transformation $L(\vec{v})$ is a map from one vector space to another over a field $\mathbb{F}$ that preserves vector addition and scalar multiplication.

- Given vector spaces V and W, a linear transformation $T: V \to W$ satisfies:
  - Additive: $T(\underbrace{u+v}_{\text{in } V}) = \underbrace{Tu + Tv}_{\text{in } W}$ $\quad \forall u,v \in V$
  - Homogeneity: $T(\underbrace{\lambda v}_{\text{in } V}) = \underbrace{\lambda(Tv)}_{\text{in } W}$ $\quad \forall \lambda \in \mathbb{F}, \forall v \in V$

In combination: $T(\alpha u + \beta v) = \alpha T(u) + \beta T(v)$

Examples:

1. $T: \mathbb{R}^3 \to \{0\}$

eg: $T(1,2,3) \to (0,0,0)$

Linear Map?

1. $T(u+v) = 0$, $T(u)+T(v) = 0+0 = 0$ ✓
2. $T(\lambda v) = 0$, $\lambda \cdot T(v) = \lambda \cdot 0 = 0$ ✓

---

2. $T: P_2 \to P_3$

$T(ax^2+bx+c) = x^3 + ax^2 + bx + c$

Linear Map? NO

1. $T((ax^2+bx+c) + (\tilde{a}x^2+\tilde{b}x+\tilde{c})) = x^3 + (a+\tilde{a})x^2 + (b+\tilde{b})x + (c+\tilde{c})$

$T(ax^2+bx+c) + T(\tilde{a}x^2+\tilde{b}x+\tilde{c}) = x^3 + ax^2 + bx + c + x^3 + \tilde{a}x^2 + \tilde{b}x + \tilde{c}$
$= \underline{2x^3} + (a+\tilde{a})x^2 + (b+\tilde{b})x + (c+\tilde{c})$

8. $T: P_2 \to P_3$

$T(ax^2+bx+c) = ax^3+bx^2+cx$

<u>Linear Map?</u>

1. $T((ax^2+bx+c)+(\tilde{a}x^2+\tilde{b}x+\tilde{c})) = (a+\tilde{a})x^3+(b+\tilde{b})x^2+(c+\tilde{c})x$ ✓

   $T(ax^2+bx+c)+T(\tilde{a}x^2+\tilde{b}x+c) = (a+\tilde{a})x^3+(b+\tilde{b})x^2+(c+\tilde{c})x$

2. $T(\lambda(ax^2+bx+c)) = a\lambda x^3+b\lambda x^2+c\lambda x$ ✓

   $\lambda(T(ax^2+bx+c)) = \lambda(ax^3+bx^2+cx) = a\lambda x^3+b\lambda x^3+c\lambda x$

(Definition) The set of all linear maps from V to W is denoted as $\mathcal{L}(V,W)$.
The set of all linear maps from V to itself is denoted as $\mathcal{L}(V)$.

Coordinate Representations

(Definition) Let B be a basis for a finite dimensional vector space V.
A vector $x \in V$ has a unique expression:

$$x = a_1 v_1 + \dots + a_n v_n.$$

Then, the coordinate vector $C_B(x)$ with respect to B.

$\hookrightarrow$ or $[x]_B$

$$\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

consequences !

- All n-dim vector spaces can be represented as $\mathbb{R}^n$.
- There exist transformations (matrices) that map 1 basis to another.
- Matrices have an identity and inverses.

Ex: P_3

- $\beta = \{x^3, x^2, x, 1\}$
- $p = 3x^2+7x+9 \in P_3 \Rightarrow [p]_\beta = \begin{bmatrix} 0 \\ 3 \\ 7 \\ 9 \end{bmatrix}$
- $\beta' = \{x^3-x^2, x^2-x, x-1, 1\}$
- $\hat{p} = 0(x^3-x^2)+3(x^2-x)+10(x-1)+19 \Rightarrow [\hat{p}]_{\beta'} = \begin{bmatrix} 0 \\ 3 \\ 10 \\ 19 \end{bmatrix}$
  $= 3x^2+7x+9$

Linear Transformations as Matrices

- For $T: \mathbb{R}^n \xrightarrow{L} \mathbb{R}^m$, the transformation can be represented as a $m \times n$ matrix.

transform the standard basis of $\mathbb{R}^n$.

$L(\vec{v}) = A\vec{v}$

Example: $T: \underset{B}{P_3} \to \underset{B'}{P_3}$ s.t

- record of where the basis vectors go.
- from here, we know where every vector goes. !

$$\left.\begin{aligned} &T(x^3) = x^3 - x^2 \\ &T(x^2) = x^2 - x \\ &T(x) = x - 1 \\ &T(1) = 1 \end{aligned}\right\} \Rightarrow \begin{array}{c|c} & \begin{array}{cccc} x^3 & x^2 & x & 1 \end{array} \\ \begin{array}{c} x^3 \\ x^2 \\ x \\ 1 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \end{array}$$

Matrix-Vector Multiplication

create a linear combination using the transformed basis vectors (which are the matrix columns). !

- When you have a matrix A that represents a linear transformation T, each column of the matrix shows where a standard basis vector gets mapped under the transformation.
- So if we have:
  - Matrix $A = [a_1\ a_2 \dots a_n]$ where each a_i is a column vector.
  - Vector $x = [x_1, x_2 \dots x_n]^T$
- Then the matrix-vector product Ax is:

  $$Ax = x_1 a_1 + x_2 a_2 + \dots + x_n a_n$$

- This is just the linear combination of the columns of A, using the components of x as the coefficients.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x \begin{bmatrix} a \\ c \end{bmatrix} + y \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} ax + by \\ ax + dy \end{bmatrix}$$

Example: $T: \mathbb{R}^4 \to \mathbb{R}^3$ — $T(51,0,1,-1) = ?$

$$\left.\begin{aligned} &- T(\overset{v_1}{1,2,3,4}) = (5,6,7) \\ &- T(\overset{v_2}{11,10,9,8}) = (2,3,1) \\ &- T(\overset{v_3}{1,5,7,2}) = (7,8,6) \\ &- T(\overset{v_4}{0,0,0,1}) = (9,1,1) \end{aligned}\right\}$$

- $\underset{\text{standard}}{\mathbb{R}^4} \xrightarrow{S} \mathbb{R}^4 \xrightarrow{T} \underset{\text{ugly}}{\mathbb{R}^3} \xrightarrow{U} \underset{\text{standard}}{\mathbb{R}^3}$ $\therefore$ $(v) = US(v)$

we only know how T works on specific vectors.

1. <u>Define $S: \mathbb{R}^4 \to \mathbb{R}^4$</u>

- $S(1,0,0,0) = \frac{25}{24}v_1 + \frac{1}{24}v_2 - \frac{1}{2}v_3 - \frac{7}{2}v_3$
- $S(0,1,0,0) = -\frac{17}{6}v_1 + \frac{1}{6}v_2 + v_3 + 8v_4$
- $S(0,0,1,0) = \frac{15}{8}v_1 - \frac{1}{8}v_2 - \frac{1}{2}v_3 - \frac{11}{2}v_4$
- $S(0,0,0,1) = v_4$

$$M(S) = \begin{pmatrix} \frac{25}{24} & -\frac{17}{6} & \frac{15}{8} & 0 \\ \frac{1}{24} & \frac{1}{6} & -\frac{1}{8} & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ -\frac{7}{2} & 8 & -\frac{11}{2} & 1 \end{pmatrix}$$

2. <u>Define $U: \mathbb{R}^3 \to \mathbb{R}^3$</u>

- $U(5,6,7)$
- $U(2,3,1)$
- $U(7,8,6)$

Vector Representation : Notation

- Let $v = (1,2,3)$
  - Then $v \in \mathbb{R}^3, \mathbb{Z}^3, \mathbb{Z}^3_{n\geq 4}, \mathbb{R}^2, \mathbb{C}^3$

- Let $w = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$
  - Then $w \in W : B_W = \{w_1, w_2, w_3\}$

$$w = 1w_1 + 2w_2 + 3w_3$$

Change of Basis

coordinate transformation

Standard basis $\rightarrow$ New Basis

- Change of basis matrix P:
  - Has basis vectors of space new basis as its columns

so it can convert ANY vector

expressed in standard coordinates

$\beta(A) = \{a_1, a_2, \dots, a_n\}$

$$P_A = \begin{pmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{pmatrix}$$

change of basis matrix

$\rightarrow$ transition matrix

coordinate transformation matrix

- $\vec{v} = P_A \cdot \vec{v}_A$ AND $\vec{v}_A = P_A^{-1} \cdot v$

Between Non-Standard Bases

- Let P_A be the matrix from $\begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix} \rightarrow \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$
  - $[x]_e = P_A [x]_A$
- Let P_B be the matrix from $\begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix} \rightarrow \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$
  - $[x]_e = P_B [x]_B$

- Then,

$$\begin{tikzcd} & e & \\ A \arrow[ur, "P_A"] \arrow[rr, shift left] & & B \arrow[ul, "P_B"'] \arrow[ll, shift left] \end{tikzcd}$$

$A \rightarrow B : P_B^{-1} P_A [x]_A$

$B \rightarrow A : P_A^{-1} P_B [x]_B$

Null space

(Definition) The null space is the set of all vectors that map to the zero vector. For a linear transformation $T: V \rightarrow W$,

$$\text{Null}(T) = \{v \in V, T(v) = 0\}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Downarrow$

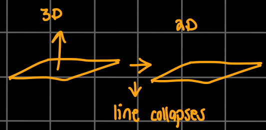
$3(x + 2y = 0)$

$-\ 3x + 4y = 0$

$$\left.\begin{array}{l} 5y = 0 \Rightarrow y = 0 \\ 3x = 0 \Rightarrow x = 0 \end{array}\right\} \text{null}(M) = \left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\}$$

Null Space Dimensionality

1. Zero-Dimensional Null Space
   - Injective (one-to-one) transformation.
   - No non-zero vectors map to zero.
   - Full rank transformation.



1 vector = line

2. One-Dimensional Null Space
   - A line of vectors maps to zero.
   - Reduces 3D $\rightarrow$ 2D.

2 lin. independent vectors forms a plane

3. Two-Dimensional Null Space
   - A plane of vectors maps to zero.
   - 3D $\rightarrow$ 1D

---

number of linearly independent rows/columns in a matrix

Rank - Nullity Theorem

the dimension of the null space

- For a linear transformation $T: V \rightarrow W$.
- $\dim(V) = \text{rank}(T) + \text{nullity}(T)$

(Theorem 3.4) Linear Map Lemma

- Suppose $v_1, \dots, v_n$ is a basis of V and $w_1, \dots, w_n \in W$. Then, $\exists!$ linear map $T: V \to W$ s.t

$Tv_k = w_k$ for each $k = 1, \dots, n$.

unique translation such that $v_1 \to w_1$, $v_2 \to w_2$, $\vdots$

(Proof):

1. Existence   Example: $x = 2v_1 + 3v_2 + 4v_3 \to 2w_1 + 3w_2 + 4w_3$

- Let $x \in V$ be an arbitrary vector. Since $\{v_1, \dots, v_n\}$ is a basis of V, $x = c_1v_1 + \dots + c_nv_n$ $\forall c_i \in \mathbb{F}$.
- Define a map $T: V \to W$ such that:

$T(x) = c_1w_1 + \dots + c_nw_n$.

- Let's prove linearity. $T(u+v) = T(u) + T(v)$ & $T(cv) = cT(v)$
  - If $u, v \in V$ then $u = a_1v_1 + \dots + a_nv_n$ and $v = c_1v_1 + \dots + c_nv_n$, then

$T(u+v) = T[(a_1+c_1)v_1 + \dots + (a_n+c_n)v_n]$
$= (a_1+c_1)w_1 + \dots + (a_n+c_n)w_n$
$= (a_1w_1 + \dots a_nw_n) + (c_1w_1 + \dots + c_nw_n)$
$= T(u) + T(v)$ ✓

  - Similarly, if $\lambda \in \mathbb{F}$ and $v = c_1v_1 + \dots + c_nv_n$, then

$T(\lambda v) = T[\lambda(c_1v_1 + \dots + c_nv_n)]$
$= T(\lambda c_1v_1 + \dots + \lambda c_nv_n)$
$= \lambda c_1w_1 + \dots + \lambda c_nw_n$
$= \lambda T(u)$ ✓

- Thus, T is a linear map from V to W.

2. Uniqueness

- Suppose $T': V \to W$ such that $T'(v_k) = w_k$ for each k.
- Let $x \in V$ s.t $x = c_1v_1 + \dots + c_nv_n$
- Then, $T'(x) = T'(c_1v_1 + \dots + c_nv_n) = c_1w_1 + \dots + c_nw_n = T(x)$
- Hence $T' = T$.

- Thus, $\exists!$ linear map $T: V \to W$ satisfying $T(v_k) = w_k$ for each $k = 1, \dots, n$

Note: It so happens that the space of linear maps $\mathcal{L}(V,W)$ is itself a vector space. $a = \{a_1, a_2, \dots, a_n\} \Rightarrow V \xrightarrow[a_i]{} W$
vector spaces

(Definition) Suppose $S, T \in \mathcal{L}(V,W)$ and $\lambda \in \mathbb{F}$.

- The sum $S+T$ and the product λT are linear maps from V to W.

1. $(S+T)(v) = S(v) + T(v)$
2. $(\lambda T)(v) = \lambda[T(v)]$

Proof:

1. WTS: $(S+T)(\alpha u + \beta v) = \alpha(S+T)(u) + \beta(S+T)(v)$

- $(S+T)(\alpha u + \beta v) = S(\alpha u + \beta v) + T(\alpha u + \beta v)$ (definition of $S+T$)
$= \alpha S(u) + \beta S(v) + \alpha T(u) + \beta T(v)$ (linearity of S & T)
$= \alpha[S(u) + T(u)] + \beta[S(v) + T(v)]$
$= \alpha(S+T)(u) + \beta(S+T)(v)$ (definition of $S+T$)

- Thus $S+T$ is linear. ■

2. WTS: $(\lambda T)(\alpha u + \beta v) = \alpha(\lambda T)(u) + \beta(\lambda T)(v)$

- $(\lambda T)(\alpha u + \beta v) = \lambda[T(\alpha u + \beta v)]$ (definition of $\lambda T(v)$)
$= \lambda[\alpha T(u) + \beta T(v)]$ (linearity of T)
$= \alpha\lambda T(u) + \beta\lambda T(v)$
$= \alpha(\lambda T)(u) + \beta(\lambda T)(u)$

- Thus λT is linear. ■

---

these operations are performed on the output of the maps.

(Theorem 3.6) Using addition and multiplication, $\mathcal{L}(V,W)$ is a vector space.

(Definition) If $T \in \mathcal{L}(U,V)$ and $S \in \mathcal{L}(V,W)$, then the product $ST \in \mathcal{L}(U,W)$ is defined by

$(ST)(u) = S(Tu)$

for all $u \in U$.

(Properties of the Products of Linear Maps)

① Associativity:

$$T_1T_2T_3 = (T_1T_2)T_3 = T_1(T_2T_3)$$

② Identity Map: $I \in \mathcal{L}(*, *)$

if $T \in \mathcal{L}(V,W)$ $\quad I \in \mathcal{L}(W,W)$

$$\underset{I \in \mathcal{L}(V,V)}{TI} = IT = T$$

③ Distributivity:

$$(S_1+S_2)T = S_1T + S_2T \qquad S_1, S_2 \in \mathcal{L}(V,W)$$

$$T \in \mathcal{L}(U,V)$$

$$R(S_1+S_2) = RS_1 + RS_2 \qquad R \in \mathcal{L}(W,X)$$

$$\longleftarrow \boxed{R} \xleftarrow{W} \boxed{S_1+S_2} \xleftarrow{V} \boxed{T} \longleftarrow$$

(Theorem 3.11) Suppose T is a linear map from V to W. Then,

$$T(0) = 0.$$

$$T(\underset{0_V}{0}) = \underset{0_W}{0}$$

(Proof)

- Let $T \in \mathcal{L}(V,W)$ and $0_V \in V$.
- By additivity:

$$T(0_V) = T(0_V + 0_V) = T(0_V) + T(0_V)$$

additive identity of V $\qquad$ additivity

- So,

$$\cancel{T(0_V)} + (-\cancel{T(0_V)}) = T(0_V) + \cancel{T(0_V)} + (\cancel{-T(0_V)})$$

$$0_W = T(0_V)$$

---

null(T) is a subset of

Ex: Suppose $D \in \mathcal{L}(P_3)$ where D is the differential operator.

$$Dp = p'$$

$$D(3x^3 + 2x^2 + x + 2) = 9x^2 + 4x + 1$$

Q: what is the null space?

- All the constants.

$$\begin{pmatrix} ? & ? & ? & 0 \\ ? & ? & ? & 0 \\ ? & ? & ? & 0 \\ ? & ? & ? & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Q: What is a matrix representation of D?

$p \in P_3 \quad p(x) = ax^3 + bx^2 + cx + d$

vector: $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$