FALL 2025

MATH 350: Mathematical Probability

August 22, Friday

probability is a function from the sample space S to the real numbers.

$P : S \to \mathbb{R}$

the function must satisfy 3 properties/axioms.

1. $P(E) \geq 0 \quad \forall E \subset S$
2. $P(S) = 1$ some outcome occurs
3. If $E \cap F = \emptyset$, then $P(E \cup F) = P(E) + P(F)$

$E \cup F$: E or F

mutually exclusive
countable additivity

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thm: $P(A') = 1 - P(A)$

proof:

- $S = A \cup A'$ and $A \cap A' = \emptyset$
- $1 = P(S) = P(A \cup A')$
  $= P(A) + P(A')$
  $1 - P(A) = P(A')$ □

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thm: $P(\emptyset) = 0$

proof:

- $\emptyset = S'$
- $P(\emptyset) = 1 - P(S) = 1 - 1 = 0$ □

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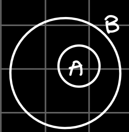
thm: If $A \subset B$, then $P(A) \leq P(B)$



proof:

- $P(B) = P(A \cup (A' \cap B))$
  $= P(A) + P(A' \cap B)$ since $A \cap (A' \cap B) = \emptyset$
  $\geq P(A)$ since $P(A' \cap B) \geq 0$

Homework

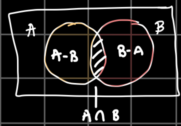
1. Prove $P(A) \leq 1 \quad \forall A \subset S$

proof:

- Since $A \subset S$, $P(A) \leq P(S) = 1$.

2. Prove $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

addition rule



proof:

- $A \cup B = A \cup (B - A)$ [disjoint union]
  $P(A \cup B) = P(A) + P(B - A)$ [additivity on disjoint sets]
- $B = (B - A) \cup (A \cap B)$ [disjoint union] partition of B
  $P(B) = P(B - A) + P(A \cap B)$ [additivity on disjoint sets]
- $P(B - A) = P(B) - P(A \cap B)$ [solve for $P(B-A)$]
- Thus, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ [substitute]

□

August 25, Monday

Enumeration

the fundamental counting principle says if one event can occur in m ways and a second can occur in n ways, the the total number of possibilities is $m \times n$ ways.

note : this only applies for independent events.

permutation : $nPk = \frac{n!}{(n-k)!} = nCk \cdot \underbrace{k!}_{\text{ordering}}$

↓ ordered

combination: $nCk = \binom{n}{k} = \frac{n!}{(n-k)!\,k!}$

↓ non-ordered

binomial coefficients : $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$

Homework

1.2-6

1 : _ _ o
2 : _ _ _ o
3 : _ _ _ _ o

winner has to win final set

- case 1 : 3-0 => $\binom{2}{2} = 1$
- case 2 : 3-1 => $\binom{3}{2} = 3$
- case 3 : 3-2 => $\binom{4}{2} = 6$

$10 \times 2 = 20$ //

↓ applies to both players

1.2-10 prove $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$

proof : WTS $\binom{n-1}{r} + \binom{n-1}{r-1} = \frac{n!}{r!(n-r)!} = \binom{n}{r}$

$$- \binom{n-1}{r} + \binom{n-1}{r-1} = \frac{(n-1)!}{r!(n-1-r)!} + \frac{(n-1)!}{(r-1)!(n-r)!}$$

$$= \frac{(n-1)!(n-r)}{r!(n-r)(n-r-1)!} + \frac{(n-1)!\,r}{r(r-1)!(n-r)!}$$

$$= \frac{(n-1)!(n-r)}{r!(n-r)!} + \frac{r(n-1)!}{r!(n-r)!}$$

$$= \frac{(n-1)!(n-r+r)}{r!(n-r)!}$$

$$= \frac{(n-1)!\,n}{r!(n-1)!} = \frac{n!}{r!(n-1)!} = \binom{n}{r}$$

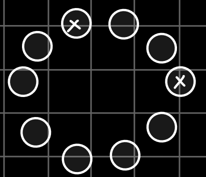
□

1.2-16

a) $P(E) = \frac{n(E)}{n(S)} = \frac{\overset{\text{white}}{\binom{19}{3}}\overset{\text{rest}}{\binom{33}{6}}}{\binom{52}{9}}$

b) $P(E) = \frac{n(E)}{n(S)} = \frac{\binom{19}{3}\binom{10}{2}\binom{7}{1}\binom{5}{1}\binom{6}{2}}{\binom{52}{9}}$

August 27, Wednesday



$7 \times 8!$

all other guests

second ex

distinguishable permutations:

- the number of ways to arrange n things where there are n_1 of the first sort, n_2 of the second and so on, is

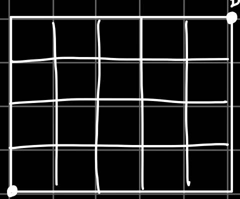
$$\frac{n!}{n_1!n_2!\dots n_r!}$$

how many ways to arrange BANANA.

$$\frac{6!}{3!2!}$$

$6!$ → total letters

$3!$ → As, $2!$ → Ns



B (5,4)

A (0,0)

how many ways from A → B using right & up?

- In general, from (a,b) to (x,y)

$$\binom{x+y}{x} \text{ or } \binom{x+y}{y}$$

since once you choose x or y, the other is fixed.

- In this case $\binom{9}{4} = \binom{9}{5} = 126$ //

stars and bars

- when: distributing identical objects into distinct categories
- examples: dice → rolls, sweets → flavors
- formula: $\binom{n+k-1}{k-1}$ n = identical objects, k = distinct categories

★★★★★★ → arrange 6 stars 4 groups

★★|★★|★|★ → choose positions for the bars → $\binom{9}{3}$

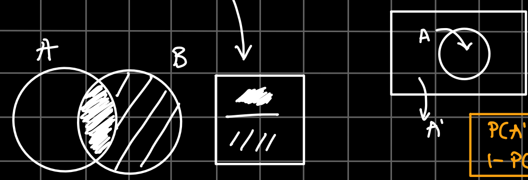
1 2 3 4 5 6 7 8 9

	Nursing	Non-nursing	Total
Males	94	1104	1198
Females	725	1682	2407
	819	2786	3605

1. $P(N) = \frac{819}{3605}$
2. $P(W|N) = \frac{725}{819}$

conditional probability: → dependent events → "without replacement"

$$P(A|B) = \frac{P(A\cap B)}{P(B)}, \quad P(B) \neq 0$$



A, B, A'

$P(A'|B)$ $1-P(A|B)$

problem:

- Roll 2 fair dice
- A = sum is odd
- B = sum is 5
- Find $P(B|A)$

solution:

- $A = \{\text{sum} = 3, 5, 7, 9, 11\}$
  ways = 2, 4, 6, 4, 2
- $|A| = 18$
- $|B| = 4$ → 1,4 or 2,3; 4,1 or 3,2
- $|A\cap B| = 4$
- $P(B|A) = \frac{P(A\cap B)}{P(A)} = \frac{4}{18}$ //

fact: $P(A|B)$ is a probability function on B.

1. $P(A|B) = \frac{P(A\cap B)}{P(B)} \geq 0$
2. $P(B|B) = \frac{P(B\cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$
3. $P((A_1 \cup A_2)|B) = \frac{P((A_1\cup A_2)\cap B)}{P(B)}$

$$= \frac{P((A_1\cap B)\cup(A_2\cap B))}{P(B)}$$

$A_1 \cup A_2 = \emptyset$ ∴ then subsets are disjoint

$$= \frac{P(A_1\cap B)}{P(B)} + \frac{P(A_2\cap B)}{P(B)}$$

$$= P(A_1|B) + P(A_2|B) \quad \square$$

multiplication rule:

- the probability that two events A and B both occur in sequence is given by

$$P(A \cap B) = P(B)\,P(A|B)$$

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problem: draw two kings without replacement

solution:

- $P(2 \text{ kings}) = P(1^{st} \text{ king}) \cdot P(2^{nd} \text{ king} \mid 1^{st} \text{ king})$

$$= \frac{4}{52} \cdot \frac{3}{51} \approx 0.0045$$

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problem: draw cards from a deck without replacement, stop when you draw 3^{rd} spade.

- Find P(6 draws).

solution:

- A = 2 spades in 1^{st} 5 draws
- B = last draw is a spade

$$\underbrace{\binom{13}{2}}_{\text{spades}}\underbrace{\binom{39}{3}}_{\text{non-spades}}$$

- $P(A) = \dfrac{\binom{13}{2}\binom{39}{3}}{\binom{52}{5}}$

- $P(B|A) = \dfrac{11}{47}$

MATH 350 HW - christopher zimbizi - August 29

1.3-7

- A = both balls are orange
- B = at least one ball is orange
- $P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{P(A)}{P(B)}$ since when both are orange, at least one is orange.

- First, number of ways to choose 2 balls such that at least one is orange.
  1. choose 2 balls: $\binom{4}{2} = 6$
  2. find ways to get both blue: $\binom{2}{2} = 1$
  3. subtract: $6-1=5$
- Now, ways to get both balls $= \binom{2}{2} = 1$
- Thus, $P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{1}{5}$ //

1.3-9

(a) $P(A_i) = \frac{\text{ways to draw ball } i \text{ on turn } i}{\text{ways to draw 4 balls}}$

① ② ③ ④

$\downarrow$ $\underbrace{\_\ \_\ \_}_{3!}$

- $\forall i \in \{1,2,3,4\}$
- There's only 1 way to place i in position i.
- There are 3! ways to arrange the remaining 3 balls in the 3 positions.
- There are 4! to arrange all 4 balls.
- Thus $P(A_i) = \frac{3!}{4!}$ //

(b) $P(A_i \cap A_j) = \frac{2!}{4!}$, $i \neq j$

① ② ③ ④

$\downarrow$ $\downarrow$ $\underbrace{\_\ \_}_{2!}$

- $\forall i \in \{1,2,3,4\}$
- There's only 1 way to place i and j in their respective positions
- There are 2! ways to arrange the remaining 2 balls in the 2 positions.
- There are 4! to arrange all 4 balls.
- Thus $P(A_i \cap A_j) = \frac{2!}{4!}$ //

(c) $P(A_i \cap A_j \cap A_k) = \frac{1!}{4!}$, $i \neq j$, $i \neq k$, $j \neq k$

① ② ③ ④

$\downarrow$ $\downarrow$ $\downarrow$ $\underbrace{\_}_{1!}$

- $\forall i \in \{1,2,3,4\}$
- There's only 1 way to place i, j and k in their respective positions
- There is 1! way to arrange the remaining ball in its position.
- There are 4! to arrange all 4 balls.
- Thus $P(A_i \cap A_j) = \frac{1!}{4!}$ //

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d) $P(\text{at least 1 match}) = P(A_1 \cup A_2 \cup A_3 \cup A_4) = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!}$

- Applying the principle of inclusion exclusion, we get $P(A_1 \cup A_2 \cup A_3 \cup A_4)$
  $= \left[\sum P(A_i) - \sum P(A_i \cap A_j) + \sum P(A_i \cap A_j \cap A_k) - \sum P(A_1 \cup A_2 \cup A_3 \cup A_4)\right]$
- Substituting the probabilities

$$P(A_1 \cup A_2 \cup A_3 \cup A_4) = \left[\binom{4}{1}\cdot\frac{1}{4}\right] - \left[\binom{4}{2}\frac{1}{4}\cdot\frac{1}{3}\right] + \left[\binom{4}{3}\frac{1}{4}\cdot\frac{1}{3}\cdot\frac{1}{2}\right] - \left[\binom{4}{4}\frac{1}{4}\cdot\frac{1}{3}\cdot\frac{1}{2}\cdot\frac{1}{1}\right]$$
$$= 1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24}$$
$$= 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} \quad \square$$

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e) Prove, given n balls, $P(\text{at least 1 match})$ is

$$P(A_1 \cup A_2 \cup \dots \cup A_n)$$
$$= 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \dots + \frac{(-1)^{n+1}}{n!}$$
$$= 1 - \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + \frac{(-1)^n}{n!}\right)$$

proof:

derangements: $D(n) = n-1\,[\underbrace{D(n-1)}_{\text{n-1 ball problem}} + \underbrace{D(n-2)}_{\text{n-2 ball problem}}]$

first ball

- $P(A_1 \cup A_2 \cup \dots \cup A_n) = P(\text{at least 1 match})$
  $= 1 - P(\text{no matches})$
- The number of assignments with no matches is the number of derangements of n items, $!n$ given by
  $$!n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + \frac{(-1)^n}{n!}\right)$$
- Since there are $n!$ total assignments
  $$P(\text{no matches}) = \frac{!n}{n!} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + \frac{(-1)^n}{n!}$$
- Thus,
  $$P(\text{at least 1 match}) = 1 - \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + \frac{(-1)^n}{n!}\right)$$
  $$= \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \dots + \frac{(-1)^{n+1}}{n!} \quad \square$$

$\frac{-1^{10}}{10!}$ $\frac{1}{}$

---

f) $\lim\limits_{n\to\infty} \left[1 - \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots \frac{(-1)^n}{n!}\right)\right]$
$\downarrow$

looks like the taylor series for e^x when $x = -1$.

so

$\lim\limits_{n\to\infty} = 1 - e^{-1} = 1 - \frac{1}{e}$ //

1.3-15

$P(R_1) = \frac{8}{15}$ $\quad P(R_2) = \frac{x}{x+9}$

$P(B_1) = \frac{7}{15}$ $\quad P(B_2) = \frac{9}{x+9}$

$$P(\text{same color}) = P(2R) + P(2B)$$
$$= [P(R_1)\cdot P(R_2)] + [P(B_1)\cdot P(B_2)]$$
$$\frac{151}{300} = \left(\frac{8}{15}\cdot\frac{x}{x+9}\right) + \left(\frac{7}{15}\cdot\frac{9}{x+9}\right)$$
$$\frac{151}{300} = \frac{8x+63}{15x+135}$$
$$151(15x+135) = 300(8x+63)$$
$$2265x - 2400x = 18\,900 - 20\,385$$
$$\therefore 135x = 1\,485$$
$$\therefore x = 11 \quad //$$

September 1, Monday

independence:

- Two events are independent if

$$P(A \cap B) = P(A) \cdot P(B)$$

- Equivalently,

$$P(A \mid B) = P(A) \quad \text{if } P(B) \neq 0$$

$$P(B \mid A) = P(B) \quad \text{if } P(A) \neq$$

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problem: is the result dependent on the treatment?

	drug	placebo	
improvement	39	25	64
no change	54	70	124
	93	95	188

solution:

- $P(D) = \frac{93}{188}$ $\quad P(I) = \frac{64}{188}$
- $P(D \cap I) = \frac{38}{188}$ $\quad P(D) \cdot P(I) = \frac{93 \cdot 64}{188^2}$
- $P(D \cap I) \neq P(D) P(I)$ $\square$

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fact: if A & B are independent, so are A & B',
A' & B
A' & B'

proof:

- Given $P(A \cap B) = P(A) \cdot P(B)$
- WWTS $P(A \cap B') = P(A) \cdot P(B')$
- $LHS = P(A \cap B') = P(A) P(B' \mid A)$ [multiplication rule]

$$= P(A)[1 - P(B \mid A)] \quad \text{[complements]}$$
$$= P(A)[1 - P(B)] \quad \text{[independence]}$$
$$= P(A) - P(A)P(B)$$
$$= P(A)[1 - P(B)]$$
$$= P(A) P(B') \quad \square$$

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problem: two coins are flipped

- A: flip1 = H
- B: flip 2 = H
- C: flip1 & 2 = HH or TT

show that these events are pairwise independent but that

$$P(A)P(B)P(C) \neq P(A \cap B \cap C)$$

solution:

- $P(A) = P(B) = P(C) = \frac{1}{2}$
- $P(A \cap B) = P(A \cap C) = P(B \cap C) = \frac{1}{4}$
- $P(A \cap B) = P(A)P(B)$
- $P(A \cap B \cap C) = \frac{1}{4}$ but $P(A)P(B)P(C) = \frac{1}{8}$

mutual independence:

- three events $A, B,$ and C are mutually indepent if they are pairwise independent and

$$P(A \cap B \cap C) =$$

MATH 350 HW - Christopher Zimbizi - September 3

1.4-5

- Given: $P(A) = 0.8$
  $P(B) = 0.5$
  $P(A \cup B) = 0.9$
- If A & B are independent, $P(A \cap B) = P(A) \cdot P(B)$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

  $$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$
  $$= 0.8 + 0.5 - 0.9$$
  $$= 0.4$$

- $P(A) \cdot P(B) = 0.8 \cdot 0.5$

  $$= 0.4.$$

- Thus, A & B are independent. ■

1.4-16

- Given:

  Ⓦ Ⓛ Ⓛ Ⓛ Ⓛ

(a) Since we have replacement, $P(\text{WIN}) = \frac{1}{5}$, $P(\text{LOSE}) = \frac{4}{5}$

- $P(1^{st} \text{ round win}) = \frac{1}{5}$

- $P(2^{nd} \text{ round win}) = \overbrace{\underbrace{\left(\tfrac{4}{5}\right)}_{\substack{\text{my}\\ \text{LOSE}}}\underbrace{\left(\tfrac{4}{5}\right)}_{\substack{\text{opp}\\ \text{LOSE}}}}^{\text{round 1}}\overbrace{\underbrace{\left(\tfrac{1}{5}\right)}_{\substack{\text{my}\\ \text{WIN}}}}^{\text{round 2}} = \left(\tfrac{4}{5}\right)^2 \cdot \tfrac{1}{5}$

- $P(3^{rd} \text{ round win}) = \overbrace{\underbrace{\left(\tfrac{4}{5}\right)}_{\substack{\text{my}\\ \text{LOSE}}}\underbrace{\left(\tfrac{4}{5}\right)}_{\substack{\text{opp}\\ \text{LOSE}}}}^{\text{round 1}}\overbrace{\underbrace{\left(\tfrac{4}{5}\right)}_{\substack{\text{my}\\ \text{LOSE}}}\underbrace{\left(\tfrac{4}{5}\right)}_{\substack{\text{opp}\\ \text{LOSE}}}}^{\text{round 2}}\overbrace{\underbrace{\tfrac{1}{5}}_{\substack{\text{me}\\ \text{WIN}}}}^{\text{round 3}} = \left(\tfrac{4}{5}\right)^4 \cdot \tfrac{1}{5}$

- $P = \frac{1}{5} + \frac{1}{5} \cdot \left(\frac{4}{5}\right)^2 + \frac{1}{5} \cdot \left(\frac{4}{5}\right)^4 + \ldots$

  geometric sum: $\sum_{k=0}^{\infty} a r^k = \frac{a}{1-r}$; $|r| < 1$ (initial term a, common ratio r)

  $$= \frac{1}{5}\left(1 + \left(\frac{4}{5}\right)^2 + \left(\frac{4}{5}\right)^4 + \ldots\right)$$
  $$= \sum_{k=0}^{\infty} \left(\frac{16}{25}\right)^k \cdot \frac{1}{5} = \frac{\frac{1}{5}}{1 - \frac{16}{25}} = \frac{\frac{1}{5}}{\frac{9}{25}} = \frac{1}{\cancel{5}_1} \cdot \frac{\cancel{25}^5}{9} = \frac{5}{9} \approx 56\%. \ ■$$

(b) Going first means you have 8 draws: 1, 8 & 5.

- If you draw WIN on any of these, you win.
- So $P(\text{you win}) = \underbrace{\frac{1}{5}}_{\text{R1 WIN}} + \underbrace{\left(\frac{\cancel{4}}{5}\right)\left(\frac{\cancel{3}}{\cancel{4}}\right)\frac{1}{\cancel{3}}}_{\text{R3 WIN}} + \underbrace{\left(\frac{\cancel{4}}{5}\right)\left(\frac{\cancel{3}}{\cancel{4}}\right)\left(\frac{\cancel{2}}{\cancel{3}}\right)\frac{1}{\cancel{2}}}_{\text{R5 WIN}}$

  $$= \frac{1}{5} + \frac{1}{5} + \frac{1}{5}$$
  $$= \frac{3}{5} = 60\%. \ ■$$

  OR $\underbrace{3}_{\text{\# of ways to win}} \times \underbrace{\frac{1}{5}}_{P(\text{WIN})} = \frac{3}{5} = 60\%$

September 3, Wednesday

Age of Driver	Probability of Accident	Fraction of Company's Insured Drivers
16 - 25	0.05	0.10
26 - 50	0.02	0.55
51 - 65	0.03	0.20
66 - 90	0.04	0.15

- What is P(Accident)?
  $\downarrow$ A

- $P(A) = P[(A \cap B_1) \cup (A \cap B_2) \cup (A \cup B_3) \cup (A \cup B_4)]$
  ↓ age 16-25 ↘ age 26-50

  $= P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + P(A|B_3)P(B_3) + P(A|B_4)P(B_4)$

  $= \sum_{i=1}^{4} P(A|B_i)P(B_i)$

  $= (.05)(.10) + (.02)(.55) + (.03)(.20) + (.04)(.15)$

  $= .028 = 2.8\%$

$\square$

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definition: partition

- A partition of a set (say S) is a collection of sets $B_1, \dots, B_K$ such that

  1. $\bigcup_{i=1}^{K} B_i = S$
  2. $B_i \cap B_j = \emptyset \; ; \; i \neq j$

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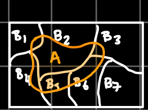
fact:

- If $B_1, \dots, B_K$ partitions S, then $\forall A \subset S$,
  $(A \cap B_1), \dots, (A \cap B_K)$
  partitions A.



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law of total probability

- if $B_1, \dots, B_K$ is a partition of S, then $\forall A \subset B$,
  $P(A) = \sum_{i=1}^{K} P(A|B_i)P(B_i)$

Bayes' Rule

- $P(B|A) = \dfrac{P(A \cap B)}{P(A)} = \dfrac{P(B) \cdot P(A|B)}{P(A)}$

  P(B|A) ↓ posterior prob.
  P(B) ↑ prior prob.
  P(A) ↓ law of total probability
  essentially reverse-engineering P(A|B) to get P(B|A)

In our example with B = "16-25", find P(B|A).

- $P(B|A) = \dfrac{P(B) \cdot P(A|B)}{P(A)} = \dfrac{(.10)(.05)}{(.028)} = 0.179 = 17.9\%$

$\square$

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example:

- A rare disease exists in 8 out of every 100,000 people.
  (8 out of every 100,000 ↑ prevalence)
- A new test has a false negative rate of 9.5% and a false positive rate of 0.9%.
- You test positive.
- What is the probability you have it.

- glossary:
  - Y = have it → $P(Y) = .00008$
  - N = don't → $P(N) = .99992$
  - + = positive
  - − = negative

- given: false negative $= P(-|Y) = .095$
  false positive $= P(+|N) = 0.09$

- formulation: $P(Y|+) = \dfrac{P(Y) \cdot \overbrace{P(+|Y)}^{1-P(-|Y)}}{P(+)}$

  $= \dfrac{P(Y) \cdot [1 - P(-|Y)]}{P(Y \cap +) + P(N \cap +)}$

  $= \dfrac{P(Y) \cdot [1 - P(-|Y)]}{P(Y) \cdot P(+|Y) + P(N) \cdot P(+|N)}$ → $1 - P(Y)$

  $= \dfrac{(.00008)(1 - .095)}{(.00008)(1 - .095) + (1 - .00008)(.009)}$

  $\approx 0.008 = 0.8\%$ //

MATH 350 HW: Due 9/5 Christopher Zimbizi

1.5-8

Supplier	Germination Rate	Fraction of Supply
A	0.85 $P(G\mid A)$	0.4 $P(A)$
B	0.75 $P(G\mid B)$	0.6 $P(B)$

(a) Find $P(G)$: probability of germination.

$$\begin{aligned} P(G) &= P(G\cap A) + P(G\cap B) \\ &= P(G\mid A)P(A) + P(G\mid B)P(A) \\ &= 0.85\cdot 0.4 + 0.75\cdot 0.6 \\ &= 0.79 \end{aligned}$$

$\square$

(b) Find $P(A\mid G)$

$$P(A\mid G) = \frac{P(A\cap G)}{P(G)} = \frac{P(A)(G\mid A)}{P(G)} = \frac{0.34}{0.79} = 0.43 \,//$$

$\square$

September 5, Friday

definition:

- A random variable (RV) X is a function from S to $\mathbb{R}$.
- The range of possible values is called the support of X.

example: flip 3 fair coins. Let X = # of heads.

- we have this distribution:

X	0	1	2	3
$P(X=x)$	$1/8$	$3/8$	$3/8$	$1/8$

random variable (X) / particular value (x)

0. TTT : $\binom{3}{0} = 1$
1. TTH : $\binom{3}{1} = 3$
2. THH : $\binom{3}{2} = 3$
3. HHH : $\binom{3}{3} = 1$

Random variables (RVs) can be discrete or continuous.

discrete: finite
continuous: support is an interval or a union of intervals

To describe a discrete RV, we typically write the support and the probability mass function (pmf), $f(x) = P(X=x)$.

- In that last example:

$$f(0) = 1/8$$
$$f(1) = 3/8$$
$$f(2) = 3/8$$
$$f(3) = 1/8$$

Every pmf satisfies 3 properties:

1. $f(x) \geq 0 \quad \forall x \in S$
2. $\sum_{x \in S} f(x) = 1 \rightarrow$ all the probabilities sum to 1
3. $P(x \in A) = \sum_{x \in A} f(x) \quad \forall x \in A.$

In the last example,

$$P(X \geq 2) = f(2) + f(3)$$
$$= \frac{3}{8} + \frac{1}{8} = \frac{1}{2}.$$

example:

- $S_X = \{1, 2, 3, \ldots, 10\}$
- $f(x) = cx$.
- Find c.

solution:

- We need
$$1 = \sum_{x=1}^{10} cx$$
$$= c \sum_{x=1}^{10} x \qquad \frac{n(n+1)}{2}$$
$$= c\left(\frac{10 \cdot 11}{2}\right)$$
$$= 55c$$
$$\therefore c = \frac{1}{55} \;//$$

September 8, Monday

definition: expected value (center of mass; weighted average of all possibilities)

- The expected value of a discrete RV is
$$E(X) = \sum_{x \in S} x \cdot f(x) = \mu$$
- It's also called the mean, denoted by μ.

example: $E(Y)$

Y	0	1	2	3	4	5
$f(y)$	$6/36$	$10/36$	$8/36$	$6/36$	$4/36$	$2/36$

- $E(Y) = 0 \cdot \frac{6}{36} + 1 \cdot \frac{10}{36} + 2 \cdot \frac{8}{36} + 3 \cdot \frac{6}{36} + 4 \cdot \frac{4}{36} + 5 \cdot \frac{2}{36}$
$$\approx 1.94 \;//$$

---

definition: expected value of a function

- $E(u(x)) = \sum_{x \in S_X} u(x) \cdot f(x)$

(u: function)

example: $E(Y^2)$

- $E(Y^2) = \sum_{S_Y} y^2 \cdot f(y)$

---

fact: expected value is a linear operator

1. $E(cX) = c \cdot E(X)$
2. $E(u(X) + v(X)) = E(u(X)) + E(v(X))$

proof:

1. $E(cX) = \sum_{x \in S} cx \cdot f(x) = c \sum_{x \in S} x \cdot f(x) = c \cdot E(x)$
2. $E(u(X) + v(X)) = \sum_{x \in S} (u(x) + v(x)) \cdot f(x)$
$$= \sum_{x \in S} u(x) \cdot f(x) + v(x) \cdot f(x)$$
$$= E(u(x)) + E(v(x))$$

---

note: expected value is not always defined

example: $f(x) = \frac{6}{\pi^2 x^2} \quad X = 1, 2, 3, \ldots$

$\frac{6}{\pi^2} \sum_{x=1}^{\infty} \frac{1}{x^2} = \frac{6}{\pi^2} \cdot \frac{\pi^2}{6} = 1 \quad \therefore f(x)$ is a pmf.

- $E(x) = x \cdot f(x) = \frac{6}{\pi^2} \cdot \sum_{x=1}^{\infty} x \cdot \frac{1}{x^2} = \frac{6}{\pi^2} \cdot \sum_{x=1}^{\infty} \frac{1}{x} = \infty \;//$

MATH 350 HW: Due 9/8 Christopher Zimbizi

1.5-8

Brands	Fraction	P(repair)
B_1	0.4	0.1
B_2	0.3	0.05
B_3	0.2	0.03
B_4	0.1	0.02

$P(B_1) = .4$
$P(B_2) = .3$
$P(B_3) = .2$
$P(B_4) = .1$

- $P(B_i \mid \text{Repair}) = \dfrac{P(B_i \cap \text{Repair})}{P(\text{Repair})} = \dfrac{P(B_i)\,P(\text{Repair} \mid B_i)}{P(\text{Repair})}$

- $P(\text{Repair}) = \sum_{i=1}^{4} P(B_i \cap \text{Repair})$

$= (.4 \times .1) + (.3 \times .05) + (.2 \times .03) + (.1 \times .02)$

$= .04 + .015 + 0.003 + .002$

$= .055$

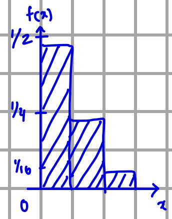
- $P(B_1 \mid \text{Repair}) = \dfrac{(.4 \times .1)}{0.55} = .072 = 7.3\%$
- $P(B_2 \mid \text{Repair}) = \dfrac{(.3 \times .05)}{0.55} = .027 = 2.7\%$
- $P(B_3 \mid \text{Repair}) = \dfrac{(.2 \times .03)}{0.55} = .0109 = 1.1\%$
- $P(B_4 \mid \text{Repair}) = \dfrac{(.1 \times .02)}{0.55} = .0036 = .36\%$

2.1-3

(c) $f(x) = c\left(\frac{1}{4}\right)^x, \quad x \in \mathbb{N}$

$1 = c \sum_{x=1}^{\infty} \left(\frac{1}{4}\right)^x$ ↙ geometric

$\frac{1}{c} = \dfrac{1/4}{1 - 1/4} = \dfrac{1/4}{3/4} = \frac{1}{3}$



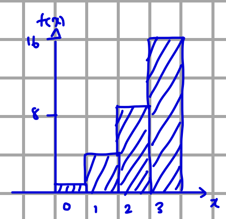
$\therefore\ c = 3 \Rightarrow f(x) = 3\left(\frac{1}{4}\right)^x \quad \forall x \in \mathbb{N}$ $\square$

(d) $f(x) = c(x+1)^2, \quad x = 0, 1, 2, 3$

$1 = c \sum_{x=0}^{3} (x+1)^2$

$\frac{1}{c} = 1 + 4 + 9 + 16$

$= 30$



$\therefore\ c = \frac{1}{30} \Rightarrow f(x) = \dfrac{(x+1)^2}{30} \quad \forall x \in [0,3]$ $\square$

2.1-7

$Y = |\max(i,j) - \min(i,j)|$

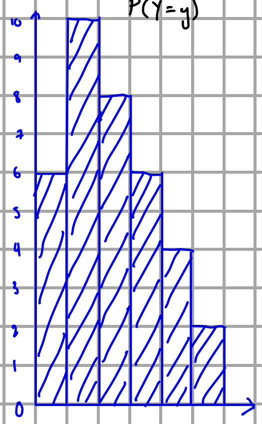
For difference d, if die 1 shows i and die 2 shows $i+d$:
1. $1 \le i \le 6$
2. $1 \le i+d \le 6$
3. $i \le 6-d$
$\Rightarrow 1 \le i \le 6-d$ possible locations for i

total $= 2(6-d)$ same for j

(c) $P(Y=0)$: $i = j$
- (1,1), (2,2), (3,3), (4,4), (5,5), (6,6)
- 6 ways $= 6/36 = 1/6$

- $P(Y=1)$: dice differ by 1.
  - (1,2), (2,3), (3,4), (4,5), (5,6)
  - $= 2 \times 5 = 10$ ways $= 10/36$ (permute)
- $P(Y=2)$: dice differ by 2.
  - (1,3), (2,4), (3,5), (4,6)
  - $2 \times 4 = 8$ ways $= 8/36$
- $P(Y=3)$: dice differ by 3
  - (1,4), (2,5), (3,6)
  - $2 \times 3 = 6$ ways $= 6/36$
- $P(Y=4)$: dice differ by 4
  - (1,5), (2,6)
  - $2 \times 2 = 4$ ways $= 4/36$
- $P(Y=5)$: dice differ by 5
  - (1,6)
  - $2 \times 1 = 2$ ways $= 2/36$

(d) $P(Y=y)$



September 10, wednesday

definition: variance
how spread out is the data from the expected value (mean)

- the variance of a RV is

$$\sigma^2 = E[(X-\mu)^2] = \sum_{x \in S_x} (x-\mu)^2 \cdot f(x)$$

deviation

---

definition: standard deviation
average amount by which a random value will differ from the mean

- the standard deviation of a RV is

$$\sigma = \sqrt{\sigma^2} = \sqrt{E[(X-\mu)^2]}$$

most results fall within 1 std. dev

---

thm: variance of a sum of independent random variables

- when X and Y are independent, then

$$\sigma^2_{X+Y} = \sigma^2_X + \sigma^2_Y$$

---

thm: variance of a scaled random variable

- If c is a constant and X is a random variable, then

$$\sigma^2_{cX} = c^2 \sigma^2_X$$

stretches/shrinks spread, variance sales by the square

---

thm: variance of the difference of independent random variables

- when X and Y are independent, then

$$\sigma^2_{X-Y} = \sigma^2_X + \sigma^2_Y$$

variance ALWAYS adds

proof:

- Consider X-Y. Rewrite as X+(-Y)
- $\sigma^2_{X-Y} = \sigma^2_X + \sigma^2_{-Y}$
- $= \sigma^2_X + (-1)^2 \sigma^2_Y$

$= \sigma^2_X + \sigma^2_Y$ □

---

thm: standard deviation of a sum of independent random variables

- when X and Y are independent, then

$$\sigma_{X+Y} = \sqrt{\sigma^2_X + \sigma^2_Y}$$

square root of variance addition

---

thm: standard deviation of a scaled random variable

- If c is a constant and X is a random variable, then

$$\sigma_{cX} = |c| \cdot \sigma_X$$

standard deviation is ALWAYS > 0.

---

thm: standard deviation of the difference of independent random variables

- when X and Y are independent, then

$$\sigma_{X-Y} = \sqrt{\sigma^2_X + \sigma^2_Y}$$

gets smaller

---

formula: variance shortcut

- For any random variable X with a well-defined mean and variance,

$$\sigma^2 = Var(X) = E(X^2) - [E(X)]^2$$

average of squares; square of average

proof:

- $\sigma^2 = E[(X-\mu)^2]$ [by definition]
  $= E[X^2 - 2\mu \cdot X + \mu^2]$ [expanding]
  $= E(X^2) - 2\mu \cdot E(X) + E(\mu^2)$ [linearity of E]
  $= E(X^2) - 2\mu^2 + \mu^2$ [$\mu = E(X)$]
  $= E(X^2) - \mu^2$
  $= E(X^2) - [E(X)]^2$ [$E(X) = \mu$]

---

definition: Bernoulli trial

- A Bernoulli trial is a probability experiment in which there are exactly two outcomes, typically called success and failure.
- The probability of success is denoted p. Failure is q.
- We can view a Bernoulli trial as a discrete RV by encoding success as 1 and failure as 0.

x	0	1
f(x)	1-p	p

---

definition: expected value of a Bernoulli trial

- The expected value of such a random value is

$$\mu = \sum x \cdot P(x)$$
$$= 0 \cdot (1-p) + 1 \cdot (p)$$
$$= p.$$

---

definition: variance of a Bernoulli trial

x	0	1
x^2	0	1
f(x)	1-p	p

- So $\sigma^2 = E(X)^2 - [E(X)]^2$
  $= p - p^2$
  $= p(1-p) = pq$

- Thus, $\sigma = \sqrt{p(1-p)} = \sqrt{pq}$

definition: binomial distribution $\binom{n}{x} p^n \cdot (1-p)^{n-x}$

- The binomial distribution, abbreviated B(n,p) or Bin(n,p), counts the number of successes in n independent Bernoulli trials, each with the same probability of success p.

---

definition: pmf of a random variable $X \sim B(n,p)$ dbinom(x,n,p)

- $$f(x) = \underbrace{\binom{n}{x}}_{\text{ways of getting x successes in n trials}} \underbrace{p^x}_{\text{successes}} \underbrace{(1-p)^{n-x}}_{\text{failures}}, \quad x = 0, 1, \dots, n$$

---

definition: expected value, variance and standard deviation of a binomial distribution.

- $\mu = np$ trials x successes
- $\sigma_X^2 = np(1-p)$
- $\sigma_X = \sqrt{np(1-p)}$

---

definition: cumulative distribution function (cdf) of B(n,p).

$$\underbrace{F(x) = P(X \leq x)}_{\text{pbinom}(x,n,p)} = \underbrace{\sum_{k=0}^{x} \binom{n}{k} p^k (1-p)^{n-k}}_{\text{sum[dbinom}(0:x,n,p)]}, \quad x = 0, 1, \dots, n$$

MATH 350 HW: Due 9/12 Christopher Zimbizi

2.2-10

- $P(\text{sum} = \{2,3,4,5,6\}) = \frac{15}{36}$
- $P(\text{sum} = \{8,9,10,11,12\}) = \frac{15}{36}$

low / high (36):

1+1	2+1	3+1	4+1	5+1	$\boxed{6+1}$
1+2	2+2	3+2	4+2	$\boxed{5+2}$	6+2
1+3	2+3	3+3	$\boxed{4+3}$	5+3	6+3
1+4	2+4	$\boxed{3+4}$	4+4	5+4	6+4
1+5	$\boxed{2+5}$	3+5	4+5	5+5	6+5
$\boxed{1+6}$	2+6	3+6	4+6	5+6	6+6

$\boxed{} \to 7$

- $P(\text{sum} = 7) = \frac{6}{36}$
- $E(X) = \sum_{x \in S} x \cdot f(x)$
- $\quad = P(\text{win}) \cdot \text{win amount} + P(\text{lose}) \cdot \text{lose amount}$

$E(X_{low}) = \left(\frac{15}{36}\right) \cdot (1) + \left(\frac{21}{36}\right) \cdot (-1)$

$= {}^{15}/_{36} - {}^{21}/_{36} = -\frac{30}{36} = -{}^{1}/_{6} \approx -\$0.1667$

high & low:

x	+1	-1
$f(x)$	$\frac{15}{36}$	$\frac{21}{36}$

$E(X_{high}) = -\$0.1667$

$E(X_7) = ({}^{6}/_{36}) \cdot (4) + ({}^{30}/_{36}) \cdot (-1)$

$= {}^{24}/_{36} - {}^{30}/_{36} = -{}^{1}/_{6} = -\$0.1667$

7:

x	+4	-1
$f(x)$	$\frac{6}{36}$	$\frac{30}{36}$

- In all 3 cases, $E(X) = -{}^{1}/_{6}$.

2.2-12

(a) Pick a random class

- X = class size of random class
- $P(X = x)$ = probability a randomly chosen class has x students

x	25	100	300
$P(X=x)$	${}^{16}/_{20}$	${}^{3}/_{20}$	${}^{1}/_{20}$

- $E(X) = 0.8(25) + 0.15(100) + 0.05(300)$
  $= 20 + 15 + 15$
  $= 50$ //

alternative

$\frac{\text{total students}}{\text{total classes}} = \frac{1000}{20}$
$= 50$ //

(b) Pick a random student out of the 1000.

- X = class size of random student
- $P(X = x)$ = probability that a student is in a class of size x

class size	# of classes	# of students
25	16	400
100	3	300
300	1	300

x	25	100	300
$P(X=x)$	$\frac{400}{1000} = 0.4$	$\frac{300}{1000} = 0.3$	$\frac{300}{1000} = 0.3$

- $E(X) = 130$ → your experience as a student
- Thus, the pmf of X is:

$$f(x) = \begin{cases} 0.4 & \text{if } x = 25, \\ 0.3 & \text{if } x = 100, \\ 0.3 & \text{if } x = 300, \\ 0 & \text{otherwise} \end{cases}$$

2.3-1

(c) $f(x) = {}^{1}/_{5}$ ← discrete uniform distribution $\quad x = 3, 5, 7, 9, 11$

```
p <- 1/5
x <- c(3,5,7,9,11)
```

① $\mu = E(X) = \sum_{x \in S} x \cdot f_x$
$= \frac{1}{5} \sum_{x \in S} x$ $\quad$ p* sum(x)
$= 7$ //

② $\sigma^2 = E(X^2) - [E(X)]^2$
$E(X^2) = \frac{1}{5} \sum_{x \in S} x^2$ $\quad$ p* sum(x^2)
$= 57$
$\sigma^2 = 57 - 49 = 8$ //

③ skewness $= E(X-\mu)^3 = E(X^3) - 3\mu\sigma^2 - \mu^3$
$= \sum (x-\mu)^3 \cdot f(x)$
$= \frac{1}{5} \sum (x-7)^3$ $\quad$ p* sum((x-7)^3)
$= 0$ (symmetric)

2.3-7

- X = randomly selected integer from $\{1, 2, \ldots, m\}$
- Find m s.t. $E(X) = Var(X)$
- $f(x) = \frac{1}{m}$ (probability of selecting 1 integer from m)
- $E(X) = \sum x \cdot f(x)$
  $= \frac{1}{m} \sum m$ → since we're just summing the integers up to m.
  $= \frac{1}{m} \cdot \frac{m(m+1)}{2}$
  $= \frac{m+1}{2}$
- $Var(X) = E(X^2) - [E(X)]^2$
  $E(X^2) = \frac{1}{m} \sum m^2$
  $= \frac{1}{\cancel{m}} \cdot \frac{\cancel{m}(m+1)(2m+1)}{6}$
  $= \frac{(m+1)(2m+1)}{6}$

$Var(X) = \frac{(m+1)(2m+1)}{6} - \left(\frac{m+1}{2}\right)^2 = \frac{(m+1)(2m+1)}{6} - \frac{(m+1)^2}{4}$

$= \frac{2(m+1)(2m+1) - 3(m+1)^2}{12}$

$= \frac{(m+1)[2(2m+1) - 3(m+1)]}{12} = \frac{(m+1)(4m+2-3m-3)}{12}$

$= \frac{(m+1)(m-1)}{12} = \frac{m^2-1}{12}$

- Now set $E(X) = Var(X)$

$\frac{m+1}{2} = \frac{m^2-1}{12}$

$2m^2 - 2 - 12m - 12 = 0$

$2m^2 - 12m - 14 = 0$

$m^2 - 6m - 7 = 0$

$m = \frac{-(-6) \pm \sqrt{36+28}}{2} = \frac{6 \pm \sqrt{64}}{2} = 3 \pm 4$

$\therefore m = 7$ or $m = -7$

- Since $m > 0$, $m = 7$.

$\square$

2.3-8

- X = larger outcome of two fair 4-sided dice rolls
- $f(x) = \frac{2x-1}{16}$, $x = 1, 2, 3, 4$ x <- 1:4

① $\mu = E(X) = \sum x \cdot f(x)$
$= \sum x \cdot \frac{2x-1}{16} = \frac{1}{16} \sum 2x^2 - x$ 1/16* sum(2*(x^2)-x))
$= 3.125$ //

② $\sigma^2 = E(X^2) - [E(X)]^2$
$= \sum x^2 \cdot \frac{2x-1}{16} - 3.125^2$
$= \frac{1}{16} \sum 2x^3 - x^2 - 3.125^2$
$= 10.625 - 3.125^2$
$= 0.859$ //

③ $\sigma = \sqrt{\sigma^2}$
$= 0.927$ //

2.4-9

(a) $X \sim B(n, p) = B(20, 0.8)$ [# of multitaskers]
binomially distributed

(b) ① $\mu = np$
$= 20 \cdot 0.8$
$= 16$ //

② $\sigma^2 = n \cdot p(1-p)$
$= 20 \cdot 0.8 \cdot 0.2$
$= 3.2$ //

③ $\sigma = \sqrt{\sigma^2}$
$= \sqrt{3.2}$
$= 1.789$ //

(c) (i) $P(X = 15) = \binom{20}{15}(0.8)^{15}(0.2)^5$ dbinom(15, 20, 0.8)
$= 0.175$ //

(ii) $P(X > 15) = \sum_{k=16}^{20} \binom{20}{k}(0.8)^k(0.2)^{20-k}$ 1- pbinom(15, 20, 0.8)
$= 0.63$ //

(iii) $P(X \leq 15) = \sum_{k=0}^{15} \binom{20}{k}(0.8)^k(0.2)^{20-k}$ pbinom(15, 20, 0.8)
$= 0.37$ //

September 12, Friday