

Wild number for spanning trees in complete and complete bipartite graphs

Aleyah Dawkins ^{*} Alexander Nappo [†] Jessie Tong [‡] Christopher Zimbizi [§]

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Abstract

Given an edge-colored graph, the wild number measures how close the graph is to having a spanning tree in every color. We investigate this parameter for complete graphs and complete bipartite graphs, with particular emphasis on minimum proper colorings and equitable 2-colorings. For complete graphs, we determine the full range of wild numbers attainable by equitable 2-colorings and show that, although every value in this range occurs, the expected wild number of a uniformly random equitable 2-coloring tends to 0 as the order of the graph grows. For minimum proper colorings of K_n , we establish a general lower bound, determine the exact wild number for $3 \leq n \leq 12$, and conjecture that the lower bound is sharp for all n . We also develop structural conditions on ideal wild sets and use them to obtain upper bounds and analyze the performance of a natural greedy algorithm. For complete bipartite graphs, we prove a sharp upper bound for equitable 2-colorings and determine the wild number exactly for minimum proper colorings. Finally, we apply these results to Cartesian products of complete graphs and complete bipartite graphs, obtaining lower bounds for the wild number under minimum proper and product colorings.

1 Introduction

Since their introduction in 1961 by Nash-Williams and Tutte [15, 18], edge-disjoint spanning trees have been instrumental in studying network reliability and performance. The existence of multiple edge-disjoint spanning trees provides independent, yet parallel, paths between vertices, allowing a network to withstand edge failures and avoid bottlenecks. These independent pathways are especially important in networks that wish to support collective communication operations, as they can be used to enhance fault tolerance while managing congestion. Independently, Nash-Williams and Tutte proved that a connected graph G has k edge-disjoint spanning trees if and only if, for any partition P of the vertices of G , there are at least $k(|P| - 1)$ crossing edges. Determining conditions that guarantee edge-disjoint spanning trees in various graphs has been studied extensively and remains an active area of research [17, 5, 12].

Edge colorings provide a natural way to organize and allocate the pathways of a network across distinct operations. Tait [16] showed that the four-color problem is equivalent to 3-edge coloring every planar 3-connected cubic graph, and edge colorings of many types have since been defined and studied [19, 10, 7, 11]. We refer the reader to a recent survey on edge colorings by Cao, Chen, Jing, Stiebitz, and Toft [4] for a more detailed overview. The applications of edge colorings are far-reaching, including scheduling, routing, and resource allocation [13, 8, 14, 3].

These two topics—edge-disjoint spanning trees as a gauge of network reliability and edge colorings as a means of organizing network operations—come together naturally in the study of the wild number of an edge-colored graph. In network terms, it is the minimum number of links that must be upgraded to carry every operation so that each operation has its own spanning tree. This idea, first introduced and explored by Anders, Foster-Greenwood, Garcia, and Krawzik [2], was inspired by the algebraic theory of generalized

^{*}Carnegie Mellon University, adawkins@andrew.cmu.edu.

[†]Villanova University

[‡]San Diego State University

[§]Lake Forest College

graph splines. See [1] for more on the connection between graphs that admit only constant splines and the wild number of a graph. The Nash-Williams–Tutte theorem can then be thought of as a characterization of when a graph has a k -edge coloring that produces wild number 0.

In this paper, we investigate the wild number of the complete graph and the complete bipartite graph for minimum proper edge colorings and equitable 2-edge colorings. Complete graphs are a natural setting for many edge coloring problems, for example, all-to-all scheduling. The complete bipartite graph has applications to scheduling as well, where the interest is the possible interactions between two groups. Minimum proper colorings offer the shortest possible conflict-free schedule for operations that cannot be performed simultaneously, and equitable 2-colorings offer insight into distributing workload evenly between two operations to prevent congestion.

We begin with preliminary definitions and results in Section 2. In Section 3, we investigate complete graphs, determining the range of possible wild numbers for equitable 2-colorings, along with the expected wild number of a uniformly random equitable 2-coloring. We also provide a lower bound on the wild number for minimum proper colorings and verify that this lower bound is sharp for $3 \leq n \leq 12$, motivating a conjecture that it is sharp for all n , as well as an analysis of the performance of a natural greedy algorithm for constructing these wild sets. We shift our focus to complete bipartite graphs in Section 4, proving a sharp upper bound on the wild number for equitable 2-colorings and determining the wild number exactly for minimum proper colorings. In Section 5, we use our results from Sections 3 and 4 on minimum proper colorings to obtain lower bounds for Cartesian products of complete graphs and of complete bipartite graphs. We conclude with some remarks in Section 6.

2 Preliminaries

Let $C = \{c_1, \dots, c_\ell\}$ be a set of colors. An ℓ -edge coloring of a graph G is a function $\gamma : E(G) \rightarrow C$ that assigns a color to each edge of G . We call the edge set of a particular color a *color class*. A connected graph G with an edge coloring γ is called an *edge-colored graph*, denoted (G, γ) . When context is clear, we refer to an edge-colored graph (G, γ) simply as G . Throughout this paper, every edge coloring is assumed to be surjective.

Let $W \subseteq E(G)$. For each color c_i , let $E(G_i) = \{e \in E(G) : \gamma(e) = c_i\}$, and let G_i be the spanning subgraph of G with edge set $E(G_i)$. Define G_i^W to be the spanning subgraph of G with edge set $E(G_i) \cup W$. If G_i^W is connected for every $c_i \in C$, then W is a *color-connecting set*. We say that an edge e is colored *wild* if $e \in W$. A color-connecting set of minimum size is an *ideal wild set*. The *wild number* of G with respect to the coloring γ is

$$\text{wld}_\gamma(G) := \min\{|W| : W \text{ is a color-connecting set for } G \text{ with coloring } \gamma\}.$$

When the edge coloring is clear from context, we abbreviate $\text{wld}_\gamma(G)$ by $\text{wld}(G)$. If $\ell = 1$, then $\text{wld}_\gamma(G) = 0$, so we assume $\ell \geq 2$ throughout. We focus our investigation on two commonly studied edge colorings, minimum proper and equitable. A *proper edge coloring* assigns a color to each edge of G such that adjacent edges receive different colors. A proper coloring of G is *minimum* if it uses exactly $\chi'(G)$ colors. An edge coloring is *equitable* if the sizes of any two color classes differ by at most one. Note that it is not required that an equitable coloring be proper. For any $1 \leq i \leq \ell$, we use the notation $\kappa(G_i)$ for the number of connected components of the monochromatic subgraph G_i . For general graphs G , we use $\kappa(G)$ to denote the number of connected components in G and $\kappa^*(G)$ to denote the number of nontrivial connected components in G , that is, components containing at least one edge.

By the definition of the wild number, for any coloring γ of an n -vertex graph G , we have that

$$0 \leq \text{wld}_\gamma(G) \leq n - 1. \tag{1}$$

The following results allow the wild number of a graph to be bounded by the number of monochromatic components.

Proposition 2.1 ([2, Proposition 5.1]). *If (G, γ) is an edge-colored graph with ℓ colors, then*

$$\max_{1 \leq i \leq \ell} (\kappa(G_i) - 1) \leq \text{wld}_\gamma(G) \leq \sum_{i=1}^{\ell} (\kappa(G_i) - 1).$$

Proposition 2.2 ([2, Proposition 5.4]). *If (G, γ) is an edge-colored graph with ℓ colors, then*

$$\text{wld}_\gamma(G) \geq \left\lceil \frac{\sum_{i=1}^{\ell} (\kappa(G_i) - 1)}{\ell - 1} \right\rceil.$$

For $\ell = 2$, the upper bound in Proposition 2.1 and the bound in Proposition 2.2 coincide, so every 2-edge-colored graph satisfies [2, Theorem 6.4(2)]

$$\text{wld}(G) = (\kappa(G_1) - 1) + (\kappa(G_2) - 1). \quad (2)$$

3 Complete Graphs

In this section, we explore the wild number of the complete graph given particular edge colorings. We first consider 2-colorings, then restrict to equitable 2-colorings before finally considering minimum proper colorings.

3.1 2-Colorings

A classical theorem of Nash-Williams and Tutte [15, 18] implies that the spanning tree packing number of K_n is $\lfloor n/2 \rfloor$. For $n \geq 4$, this yields at least 2 edge-disjoint spanning trees in K_n ; call them T_1 and T_2 . Let $\gamma : E(K_n) \rightarrow \{c_1, c_2\}$ be an edge coloring such that $E(T_1) \subseteq E(G_1)$ and $E(T_2) \subseteq E(G_2)$, where the remaining edges of K_n are colored arbitrarily. Then both G_1 and G_2 are connected spanning subgraphs. Thus, by (2), $\text{wld}_\gamma(K_n) = (1 - 1) + (1 - 1) = 0$.

We now construct a 2-coloring producing a color class with r components.

Construction 3.1 (Clique-fragmentation coloring). Let $n > r \geq 2$. Fix a partition of n into r positive integers $s_1 + s_2 + \dots + s_r = n$. Partition $V(K_n)$ into disjoint, nonempty sets $V_1, \dots, V_r$ with $|V_i| = s_i$. Define a 2-coloring such that:

- $G_1 = \bigcup_{i=1}^r K_{s_i}$ (a disjoint union of cliques),
- $G_2 = K_{s_1, s_2, \dots, s_r}$ (the complement, a complete multipartite graph).

The condition $r \geq 2$ ensures $E(G_2)$ is nonempty, and $n > r$ ensures $E(G_1)$ is nonempty, guaranteeing surjectivity. An example of such a construction is shown in Fig. 3.1 where $n = 12$, $r = 3$, and $s_i = 4$ for all i .

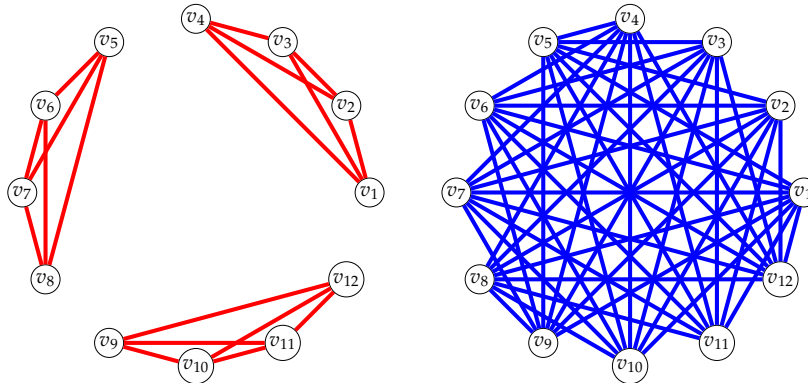


Figure 3.1: An example of Construction 3.1.

The following lemma shows that the wild number of K_n with a clique-fragmentation coloring is determined by the number of parts in the vertex partition of K_n .

Lemma 3.2. *For any clique-fragmentation coloring with r parts, we have*

$$\text{wld}(K_n) = r - 1.$$

Proof. By construction, G_1 consists of r vertex-disjoint cliques, so $\kappa(G_1) = r$. Since $r \geq 2$ and $s_i \geq 1$ for all i , the complete multipartite graph G_2 is connected. Applying (2), we have

$$\text{wld}(K_n) = \kappa(G_1) + \kappa(G_2) - 2 = r + 1 - 2 = r - 1. \quad \square$$

Theorem 3.3. *Let $n \geq 4$. For any 2-coloring of K_n ,*

$$0 \leq \text{wld}(K_n) \leq n - 2.$$

Moreover, every integer in this range is attainable by some 2-coloring.

Proof. The surjectivity of the coloring guarantees that $\kappa(G_1) \geq 1$ and $\kappa(G_2) \geq 1$. For any graph H on $n \geq 2$ vertices, at least one of H or its complement $\overline{H}$ is connected. Since G_1 and G_2 are complements of each other, we then have $\kappa(G_1) = 1$ or $\kappa(G_2) = 1$, say $\kappa(G_1) = 1$. A graph on n vertices containing at least one edge has at most $n - 1$ connected components, so

$$\kappa(G_1) + \kappa(G_2) \leq 1 + (n - 1) = n.$$

From Proposition 2.1, it follows that $0 \leq \text{wld}(K_n) \leq n - 2$. Since $n \geq 4$, the attainability of the wild number 0 is shown by the coloring discussed at the beginning of this subsection. For $1 \leq k \leq n - 2$, choose $r = k + 1$ in Construction 3.1. Then, by Lemma 3.2, we have $\text{wld}(K_n) = r - 1 = k$, verifying the attainability of every wild number in the range $\{1, \dots, n - 2\}$. $\square$

3.2 Equitable 2-Colorings

We determine the set of wild numbers attainable by equitable 2-colorings of K_n . We begin by constructing an equitable 2-coloring that can be defined to achieve any number in the full range of possible wild numbers.

Construction 3.4. Let $n \geq 4$ and let $m \in \{\lfloor n(n - 1)/4 \rfloor, \lceil n(n - 1)/4 \rceil\}$. Set $t = \left\lceil \frac{1 + \sqrt{1 + 8m}}{2} \right\rceil$. Fix m and let $1 \leq k \leq n - t$. We construct an equitable 2-coloring such that there is a color class containing m edges that form $k + 1$ components consisting of one component of size $n - k$ and the remaining components consisting of isolated vertices.

Partition $V(K_n)$ into two sets U_1 and V_1 such that $|U_1| = k$ and $|V_1| = n - k$. Choose a spanning tree of $K_n[V_1]$, and then add further edges of $K_n[V_1]$ until the subgraph has m edges. Existence of m edges is guaranteed since $k \leq n - t$ gives $n - k \geq t$, and so $\binom{n-k}{2} \geq \binom{t}{2} \geq m$. Connectedness is guaranteed, since $n \geq 4$, by $m \geq \lfloor n(n - 1)/4 \rfloor \geq n - 2 \geq n - k - 1$.

Let this set of m edges be $M \subset E(K_n)$. Define a 2-coloring γ such that:

- G_1 is the graph with vertex set $V(K_n)$ and edge set M .
- G_2 is the graph with vertex set $V(K_n)$ and edge set $E(K_n) \setminus M$.

Note that $\kappa(G_1) = k + 1$ and $\kappa(G_2) = 1$ since G_2 contains every edge between U_1 and V_1 . An example of Construction 3.4 is shown in Fig. 3.2 for $n = 12$, $m = 33$, and $k = n - t = 3$.

Lemma 3.5. *Let $n \geq 4$ and let $m \in \{\lfloor n(n - 1)/4 \rfloor, \lceil n(n - 1)/4 \rceil\}$. Fix m and let $1 \leq k \leq n - \left\lceil \frac{1 + \sqrt{1 + 8m}}{2} \right\rceil$. Construction 3.4 defines an equitable 2-coloring ε such that*

$$\text{wld}_\varepsilon(K_n) = k.$$

Proof. By construction, G_1 consists of $k + 1$ components and G_2 is connected. From (2), as ε is a 2-coloring, it follows that $\text{wld}_\varepsilon(K_n) = \kappa(G_1) + \kappa(G_2) - 2 = (k + 1) + 1 - 2 = k$. $\square$

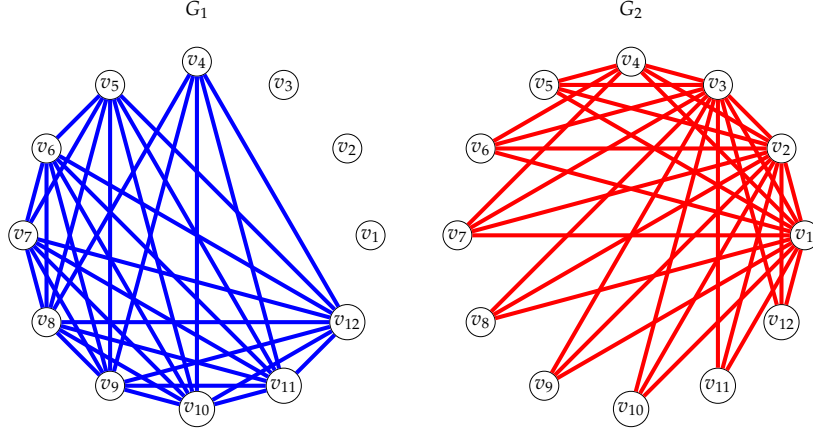


Figure 3.2: Construction 3.4, with $n = 12$ and $m = 33$, attaining the upper bound for equitable 2-colorings: $\text{wld}_e(K_{12}) = 12 - \lceil (1 + \sqrt{1 + 8 \cdot 33})/2 \rceil = 3$.

Theorem 3.6. *Let $n \geq 4$. For any equitable 2-coloring of the edges of K_n , if both color classes are connected, then*

$$\text{wld}(K_n) = 0.$$

Otherwise, let $m \in \{\lfloor n(n-1)/4 \rfloor, \lceil n(n-1)/4 \rceil\}$ be the number of edges in a disconnected color class. Then

$$1 \leq \text{wld}(K_n) \leq n - \left\lceil \frac{1 + \sqrt{1 + 8m}}{2} \right\rceil.$$

Moreover, every integer from 0 to $n - \left\lceil \frac{1 + \sqrt{1 + 8m}}{2} \right\rceil$ is attainable by some equitable 2-coloring.

Proof. By Proposition 2.1, we have

$$\text{wld}(K_n) \leq \kappa(G_1) + \kappa(G_2) - 2. \quad (3)$$

Since $G_2 = \overline{G_1}$, at most one of $\{G_1, G_2\}$ is disconnected. If both are connected, then (3) gives $\text{wld}(K_n) = 0$. For the remainder of the proof, we may assume without loss of generality that $\kappa(G_1) > 1$ and $\kappa(G_2) = 1$. Set $m = |E(G_1)|$; then $m \in \{\lfloor n(n-1)/4 \rfloor, \lceil n(n-1)/4 \rceil\}$ and $|E(G_2)| = \binom{n}{2} - m$. Now we bound the number of components in G_1 .

Let $\kappa(G_1) = q$ with components of size $a_1, a_2, \dots, a_q$ where $\sum_{i=1}^q a_i = n$ and each $a_i > 0$. The maximum number of edges in a component on a_i vertices is $\binom{a_i}{2}$, so

$$m \leq \sum_{i=1}^q \binom{a_i}{2}.$$

Since $x \mapsto \binom{x}{2}$ is convex, this sum is maximized when the component sizes are as unequal as possible, namely when one component contains $n - q + 1$ vertices and the remaining $q - 1$ components are isolated vertices. Thus,

$$m \leq \sum_{i=1}^q \binom{a_i}{2} \leq \binom{n - q + 1}{2}.$$

Let $t = \min\{s : \binom{s}{2} \geq m\}$. Then $t \leq n - q + 1$ and $q \leq n - t + 1$. Since $t = \min\{s : \binom{s}{2} \geq m\}$, we have $\binom{t}{2} = \frac{t(t-1)}{2} \geq m$ and thus $t \geq \frac{1 + \sqrt{1 + 8m}}{2}$. By minimality, it follows that $t = \left\lceil \frac{1 + \sqrt{1 + 8m}}{2} \right\rceil$. From (3), we have

$$\text{wld}(K_n) \leq q - 1 \leq n - t = n - \left\lceil \frac{1 + \sqrt{1 + 8m}}{2} \right\rceil.$$

Recall that the spanning tree packing number of K_n is $\lfloor n/2 \rfloor$, so for $n \geq 4$ there are at least two edge-disjoint spanning trees in K_n . Assigning a spanning tree to each color class and partitioning the remaining edges in an equitable way verifies the attainability of wild number 0. For all $1 \leq k \leq n - \left\lfloor \frac{1+\sqrt{1+8m}}{2} \right\rfloor$, Construction 3.4 and Lemma 3.5 ensure that $\text{wld}_\varepsilon(K_n) = k$, verifying the attainability of every wild number in the range of $\{1, \dots, n - \left\lfloor \frac{1+\sqrt{1+8m}}{2} \right\rfloor\}$. $\square$

Note that for all $n \geq 4$, we can simplify the bound to $0 \leq \text{wld}(K_n) \leq (1 - \sqrt{2}/2)n + 3/2$.

3.2.1 Expected Wild Number

As shown above, every wild number in the attainable range for an equitable 2-coloring of K_n occurs. However, the expected wild number for a uniformly random equitable 2-coloring of K_4 is only $2/5$. We show that, as n grows, the expected wild number of a uniformly random equitable 2-coloring of K_n tends to 0. The following lemma is an immediate consequence of Gilbert's bounds on the connectivity probability.

Lemma 3.7 ([9]). *Let G be the random graph on n vertices where every edge occurs independently with probability $1/2$. Then $\Pr[G \text{ is disconnected}] = O(n2^{-n})$.*

Corollary 3.8. *Let $N = \binom{n}{2}$ and let $M \in \{\lfloor N/2 \rfloor, \lceil N/2 \rceil\}$. If $G_{n,M}$ is chosen uniformly from all graphs on n vertices with M edges, then*

$$\Pr[G_{n,M} \text{ is disconnected}] = o(1/n).$$

Proof. Let G be the random graph on n vertices where every edge occurs independently with probability $1/2$. Conditional on $|E(G)| = M$, the graph G is uniformly distributed over all graphs on n vertices with exactly M edges. It follows that

$$\begin{aligned} \Pr[G_{n,M} \text{ is disconnected}] &= \Pr[G \text{ is disconnected} \mid |E(G)| = M] \\ &\leq \frac{\Pr[G \text{ is disconnected}]}{\Pr[|E(G)| = M]} \\ &= \frac{\Pr[G \text{ is disconnected}]}{2^{-N} \binom{N}{M}} \end{aligned}$$

From Lemma 3.7, we have $\Pr[G \text{ is disconnected}] = O(n2^{-n})$. Applying Stirling's approximation to the central binomial coefficient, we have

$$2^{-N} \binom{N}{M} = \Theta(N^{-1/2}) = \Theta(1/n)$$

since $|M - N/2| = O(\sqrt{N})$. Therefore

$$\Pr[G_{n,M} \text{ is disconnected}] = \frac{O(n2^{-n})}{\Theta(1/n)} = O(n^2 2^{-n}) = o(1/n). \quad \square$$

Theorem 3.9. *Let ε be a uniformly random equitable 2-coloring of K_n . Then $\mathbb{E}[\text{wld}_\varepsilon(K_n)] \rightarrow 0$ as $n \rightarrow \infty$.*

Proof. Let $N = \binom{n}{2}$. Since ε is equitable, for $i = 1, 2$, we have $|E(G_i)| \in \{\lfloor N/2 \rfloor, \lceil N/2 \rceil\}$. From (2), the wild number of K_n with ε is

$$\text{wld}(K_n) = (\kappa(G_1) - 1) + (\kappa(G_2) - 1).$$

The expected wild number of K_n is then

$$\mathbb{E}[\text{wld}(K_n)] = \mathbb{E}[\kappa(G_1) - 1] + \mathbb{E}[\kappa(G_2) - 1].$$

Conditional on its number of edges, each G_i is uniformly distributed over all graphs on n vertices with that many edges. Since a graph on n vertices can have at most n components, for $i = 1, 2$, we then have

$$\mathbb{E}[\kappa(G_i) - 1] \leq (n - 1) \Pr[G_i \text{ is disconnected}] = o(1)$$

by Corollary 3.8. Thus,

$$\mathbb{E}[\text{wld}(K_n)] = \mathbb{E}[\kappa(G_1) - 1] + \mathbb{E}[\kappa(G_2) - 1] = o(1) \rightarrow 0. \quad \square$$

3.3 Minimum Proper Colorings

When n is odd, $\chi'(K_n) = n$ and every color class is a near-perfect matching of size $(n-1)/2$; when n is even, $\chi'(K_n) = n-1$ and every color class is a perfect matching of size $n/2$. For a color c_i , we write M_i for its monochromatic matching. For odd n , we denote the isolated vertex in color c_i by v_i . Note that each color misses exactly one vertex, and each vertex misses exactly one color, so $c_i \mapsto v_i$ is a bijection between colors and vertices; in particular, the v_i are distinct.

For minimum proper colorings of K_n , ideal wild sets satisfy several structural constraints that will be used throughout this section.

Lemma 3.10. *Let $n \geq 3$ and let ρ be a minimum proper coloring of K_n . Every color-connecting set is incident to at least $n-1$ vertices, and to all n vertices when n is odd. Moreover, every ideal wild set is a forest in which every nontrivial component has at least three vertices.*

Proof. First, any color-connecting set must be incident to at least $n-1$ vertices. Indeed, if two vertices are not incident to any wild edge, then the unique edge joining them is isolated in its color class, preventing that color class from becoming connected. When n is odd, each color class is a near-perfect matching with a unique isolated vertex. Consequently, every vertex must be incident to a wild edge, as an isolated vertex would remain isolated in its missing color.

Minimality of an ideal wild set further implies that every ideal wild set is a forest. If an edge e lies on a cycle of a color-connecting set W , then the endpoints of e remain joined by a path in $W - e$, thus G_i^{W-e} would remain connected for all i implying $W - e$ is a color-connecting set. By the same argument as above, no component formed by the edges of an ideal wild set can be a single edge as the edge would remain isolated in its own color class. Thus, every nontrivial component contains at least three vertices. $\square$

3.3.1 Lower Bound

The properties of ideal wild sets discussed above provide a natural lower bound on the wild number.

Theorem 3.11. *For any minimum proper coloring ρ of K_n with $n \geq 3$, the wild number satisfies*

$$\text{wld}_\rho(K_n) \geq \left\lfloor \frac{4\lceil n/2 \rceil}{3} \right\rfloor.$$

Proof. Let W be an ideal wild set. Let s be the number of vertices incident to an edge in W . By Lemma 3.10, we have $s \geq n-1$ if n is even and $s = n$ if n is odd. In all cases, $s \geq 2\lceil n/2 \rceil - 1$. Further, we have that W is a forest on s vertices. Let t be the number of tree components of W . Since each component contains at least 3 vertices, by Lemma 3.10, we have $t \leq \lfloor s/3 \rfloor$ so $|W| = s - t \geq s - \lfloor s/3 \rfloor$. Note that the function $f(x) = x - \lfloor x/3 \rfloor$ is non-decreasing on the integers. Since $s \geq 2\lceil n/2 \rceil - 1$, we have

$$|W| \geq (2\lceil n/2 \rceil - 1) - \left\lfloor \frac{2\lceil n/2 \rceil - 1}{3} \right\rfloor.$$

Setting $k = \lceil n/2 \rceil$ and writing $k = 3q + r$ with $r \in \{0, 1, 2\}$, both $2k - 1 - \lfloor (2k - 1)/3 \rfloor$ and $\lfloor 4k/3 \rfloor$ equal $4q + r$. Therefore, $|W| \geq \left\lfloor \frac{4\lceil n/2 \rceil}{3} \right\rfloor$ and so

$$\text{wld}_\rho(K_n) \geq \left\lfloor \frac{4\lceil n/2 \rceil}{3} \right\rfloor. \quad \square$$

3.3.2 Structure of Ideal Wild Sets

Recall that v_i is the vertex isolated in color c_i . We call a set $T = \{v_i, v_j, v_k\} \subseteq V(K_n)$ a *valid triple* if $\rho(v_j v_k) \neq c_i$, $\rho(v_i v_k) \neq c_j$, and $\rho(v_i v_j) \neq c_k$. This condition is necessary for T to occur as a component of a color-connecting set, since otherwise there would be no edge of color c_i , c_j , or c_k connecting T to the remaining vertices. We call T *invalid for c_i* if T is not a valid triple because $\rho(v_j v_k) = c_i$, or simply *invalid* when color does not matter.

Lemma 3.12. *Let ρ be a minimum proper coloring of K_n . For odd $n \geq 7$, the graph K_n contains a valid triple.*

Proof. Each edge $xy \in M_i$ witnesses the invalid triple $\{v_i, x, y\}$. Conversely, every invalid triple is witnessed by at least one of the edges between its vertices.

An edge witnesses exactly one invalid triple, since its color c_i determines a unique vertex v_i . Therefore, the number of invalid triples is at most $|E(K_n)| = \binom{n}{2}$. Since $n \geq 7$, the number of total triples, $\binom{n}{3}$, exceeds $\binom{n}{2}$. Hence not every triple is invalid, and a valid triple exists. $\square$

When n is odd, Lemma 3.10 implies that any ideal wild set must be a spanning forest where every component contains at least 3 vertices. We call the vertex sets of these components *blocks*.

A forest of blocks fails to be color-connecting when a nonempty union of blocks is closed under some color class and is, therefore, disconnected from the remaining blocks. We isolate this obstruction in the following definition.

Definition 3.13. Let ρ be a minimum proper coloring of K_n with n odd, and let $P = \{B_1, \dots, B_t\}$ be a partition of $V(K_n)$ into blocks where $|B_j| \geq 3$ for $1 \leq j \leq t$. A *seal* is a pair $(c_i, \mathcal{U})$, where c_i is a color and $\mathcal{U} \subseteq P$ is a nonempty set of blocks whose vertex union $U = \bigcup_{B \in \mathcal{U}} B$ satisfies:

- (i) $v_i \notin U$, and
- (ii) every edge of M_i with an endpoint in U has both endpoints in U ; equivalently, M_i restricted to U is a perfect matching of U .

We call P *seal-free* if it admits no seal. Since a perfect matching requires an even number of vertices, $|U|$ is even for every seal; in particular, a single odd block cannot by itself form a seal. Up to relabeling, the two possible seals consisting of two triples are shown in Fig. 3.3.

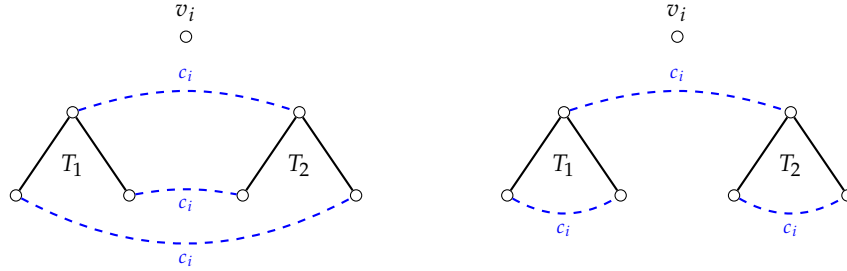


Figure 3.3: The two seals, up to relabeling, of two triples for a color c_i .

Proposition 3.14. Let ρ be a minimum proper coloring of K_n with n odd and let $P = \{B_1, \dots, B_t\}$ be a partition of $V(K_n)$ into blocks where $|B_j| \geq 3$ for $1 \leq j \leq t$. For each j , let T_j be a spanning tree of $K_n[B_j]$, and let $W = \bigcup_{j=1}^t E(T_j)$. Then W is color-connecting if and only if P is seal-free.

Proof. First, suppose P has a seal $(c_i, \mathcal{U})$, with $U = \bigcup_{B \in \mathcal{U}} B$ and $\bar{U} = V(K_n) - U$. We claim no edge of G_i^W , whose edge set is $M_i \cup W$, joins U to its complement $\bar{U}$. The blocks of a partition are pairwise disjoint, so every block lies entirely in U or entirely in $\bar{U}$. A wild edge $e \in W$ has both endpoints in a single block, and that block lies on one side, so no wild edge crosses the cut $(U, \bar{U})$; by condition (ii) of Definition 3.13, no edge of M_i crosses either. Both sides are nonempty; $U \neq \emptyset$ since $\mathcal{U} \neq \emptyset$ and $v_i \in \bar{U}$ since $v_i \notin U$. So G_i^W is disconnected and W is not color-connecting.

Conversely, suppose W is not color-connecting, say G_i^W is disconnected for the color c_i . Each block of P is connected by its wild spanning tree, so every component of G_i^W is a union of whole blocks. Hence the blocks of P split into two nonempty sets $\mathcal{A}$ and $\mathcal{B}$ with no edge of M_i between $A = \bigcup \mathcal{A}$ and $B = \bigcup \mathcal{B}$. Note that M_i is a near-perfect matching missing exactly the vertex v_i , which lies in either A or B , say $v_i \in A$. Then every vertex of B is matched by M_i , and since no edge colored c_i has exactly one endpoint in B , it has both endpoints in B . Thus M_i restricted to B is a perfect matching, and $(c_i, \mathcal{B})$ is a seal. Note that neither direction referred to the choice of spanning trees inside the blocks, so color-connectivity depends only on the partition. $\square$

Lemma 3.15. *Let ρ be a minimum proper coloring of K_n with $n \geq 7$ odd, and let $A \subseteq V(K_n)$ where $|A| = 4$. If A is a block of a partition P , then $(c_i, \{A\})$ is not a seal for any i if and only if $K_n[A]$ is a rainbow K_4 .*

Proof. Since ρ is proper, if some color c_i appears twice in $K_n[A]$, the two edges colored c_i are disjoint and cover A ; hence $v_i \notin A$ and $(c_i, \{A\})$ is a seal. Conversely, since $|A| = 4$, a seal $(c_i, \{A\})$ is a pair of disjoint edges colored c_i between the vertices of A , so c_i is a repeated color. Thus, a seal $(c_i, \{A\})$ exists for some c_i if and only if $K_n[A]$ is not rainbow. $\square$

Lemma 3.16. *Let ρ be a minimum proper coloring of K_n with $n \geq 7$ odd, and let P be a partition of $V(K_n)$ into blocks, one of which is a triple T . Let $\mathcal{U} = P \setminus \{T\}$. Then $(c_i, \mathcal{U})$ is a seal if and only if T is invalid for c_i .*

Proof. Suppose $(c_i, \mathcal{U})$ is a seal. By definition, $v_i \notin U$, so $v_i \in T$. Since $|T| = 3$, the remaining set $U = V(K_n) \setminus T$ contains $n - 3$ vertices. Since M_i restricted to U is a perfect matching of U , it uses $(n - 3)/2$ edges of M_i , so exactly $(n - 1)/2 - (n - 3)/2 = 1$ edge of M_i remains. Since every vertex of U is already matched, this edge has no endpoint in U . Thus it must have both endpoints in $T \setminus \{v_i\}$. It follows that T contains the isolated vertex v_i and its other two vertices are joined by an edge colored c_i , making T invalid for c_i .

Conversely, if T is invalid for c_i , then $T = \{v_i, a, b\}$ with $\rho(ab) = c_i$. The remaining $(n - 1)/2 - 1 = (n - 3)/2$ edges of M_i cover the remaining $n - 3$ vertices of the graph, which are $V(K_n) \setminus T = U$. Those edges form a perfect matching of U . Thus, $(c_i, \mathcal{U})$ is a seal. $\square$

3.3.3 Small Cases

The following lemma allows us to bound the wild number of an even complete graph by an odd complete graph.

Lemma 3.17. *Let $k \geq 2$, let ρ be a minimum proper coloring of K_{2k} , and let $v \in V(K_{2k})$. Define*

$$\rho_v := \rho|_{K_{2k}-v}.$$

Then ρ_v is a minimum proper coloring of K_{2k-1} and

$$\text{wld}_\rho(K_{2k}) \leq \text{wld}_{\rho_v}(K_{2k-1}).$$

Proof. The coloring ρ uses $2k - 1$ colors, and each color class is a perfect matching of size k . Deleting v removes exactly one edge from each color class. Thus every color class of ρ_v is a near-perfect matching of size $k - 1$ and ρ_v is a proper coloring of K_{2k-1} using $2k - 1$ colors, and so is minimum as $\chi'(K_{2k-1}) = 2k - 1$.

Let $W \subseteq E(K_{2k} - v)$ be color-connecting under ρ_v . Fix a color c_i and let e_i be the unique edge of M_i incident to v . The graph $(M_i - e_i) \cup W$ is connected on $V(K_{2k} - v)$ and e_i joins v to a vertex of $(M_i - e_i) \cup W$. Hence $M_i \cup W$ is connected on $V(K_{2k})$.

This holds for every color c_i , so W is also color-connecting in K_{2k} . Therefore,

$$\text{wld}_\rho(K_{2k}) \leq \text{wld}_{\rho_v}(K_{2k-1}). \quad \square$$

Lemma 3.18. *Let ρ be a minimum proper coloring of K_7 . Suppose that T is a valid triple and that B induces a rainbow K_4 and is disjoint from T . Then the union of the spanning trees of $K_7[T]$ and $K_7[B]$ is a color-connecting set of size 5.*

Proof. Note that $P = \{T, B\}$ is a block partition of $V(K_7)$ and W , consisting of a spanning tree of each block, has $|W| = (3 - 1) + (4 - 1) = 5$. By Proposition 3.14, W is color-connecting if the partition P admits no seals. Recall that a seal perfectly matches its vertices, requiring an even number of vertices. The only nonempty subset of P whose union has even size is $\{B\}$. By Lemma 3.15, $(c_i, \{B\})$ is a seal for some c_i if and only if $K_7[B]$ is not rainbow. From our assumptions, we have that $K_7[B]$ is rainbow, so no seal has the form $(c_i, \{B\})$. Thus, W is color-connecting. $\square$

Lemma 3.19. *Let ρ be a minimum proper coloring of K_9 . Then we can partition $V(K_9)$ into three valid triples.*

Proof. By Lemma 3.12, there is a valid triple; relabel so that it is $T = \{v_1, v_2, v_3\}$. Each edge $v_j v_k$ of T is not colored c_j or c_k , since v_j and v_k are isolated in those colors, and is not colored c_i , since T is valid. As ρ is proper, the three edges of T receive distinct colors, which we relabel c_4, c_5, c_6 . It remains to partition the

remaining six vertices into two valid triples. There are $\binom{6}{3}/2 = 10$ ways to partition the remaining six vertices into two unordered triples.

We now bound the number of partitions that do not produce the remaining two valid triples. Since $v_1, v_2, v_3 \in T$, no invalid triple among the remaining six vertices can arise from colors c_1, c_2, c_3 . For $\{c_4, c_5, c_6\}$, one matching edge lies inside T , and the third vertex of T is matched outside T , leaving two matching edges among the remaining six vertices. Thus, at most 2 remaining edges of each color can contribute invalid triples. For $\{c_7, c_8, c_9\}$, no matching edge lies inside T , so the three vertices of T are matched to three vertices outside T , leaving one matching edge among the remaining six vertices. Thus, at most 1 remaining edge of each color can contribute invalid triples. Therefore, among the remaining six vertices, there are at most $3 \cdot 2 + 3 \cdot 1 = 9$ invalid triples.

Each invalid triple determines at most one partition of the remaining six vertices that does not produce the two remaining valid triples, namely the partition consisting of that triple and its complement. Therefore, at most 9 of the 10 partitions are invalid. Hence at least one partition of the remaining six vertices consists of two valid triples. T together with these two valid triples gives a partition of $V(K_9)$ into three valid triples. $\square$

Lemma 3.20. *Let ρ be a minimum proper coloring of K_9 . Suppose that $T_1, T_2, T_3 \subseteq V(K_9)$ are valid triples such that T_1, T_2, T_3 are pairwise disjoint. Then the union of the spanning trees of $K_9[T_1]$, $K_9[T_2]$, and $K_9[T_3]$ is a color-connecting set of size 6.*

Proof. Note that $P = \{T_1, T_2, T_3\}$ is a block partition of $V(K_9)$, and by Proposition 3.14, it suffices to rule out the existence of seals. The union of blocks in a seal contains an even number of vertices, so $\mathcal{U}$ must consist of exactly two triples. By Lemma 3.16, a seal $(c_i, P \setminus \{T_j\})$ exists exactly when T_j is invalid for c_i . Let W be the union of the spanning trees of $K_9[T_1]$, $K_9[T_2]$, and $K_9[T_3]$. All three triples are valid, so P is seal-free and W is color-connecting with $|W| = 3(3 - 1) = 6$. $\square$

Lemma 3.21. *Let ρ be a minimum proper coloring of K_{11} . Then we can partition $V(K_{11})$ into a valid triple T and two blocks that each induce a rainbow K_4 .*

Proof. By Lemma 3.12, there is a valid triple; relabel so that it is $T = \{v_1, v_2, v_3\}$. Each edge $v_j v_k$ of T is not colored c_j or c_k , since v_j and v_k are isolated in those colors, and is not colored c_i , since T is valid. As ρ is proper, the three edges of T receive distinct colors, which we relabel c_4, c_5, c_6 . It remains to partition the remaining eight vertices into two sets, each inducing a rainbow K_4 . There are $\binom{8}{4}/2 = 35$ partitions of these eight vertices into two sets of size 4.

For the colors $\{c_1, c_2, c_3\}$, one vertex of T is missed, while the other two vertices of T are matched to vertices outside T , since T is valid. Thus, two edges of each such color are incident to T , leaving $5 - 2 = 3$ edges among the remaining eight vertices. For $\{c_4, c_5, c_6\}$, one edge of each color lies inside T , while the remaining vertex of T is matched to a vertex outside T . Thus, again, two edges of each such color are incident to T , leaving $5 - 2 = 3$ edges among the remaining eight vertices. For $\{c_7, \dots, c_{11}\}$, no edge lies inside T and no vertex of T is missed, so all three vertices of T are matched to vertices outside T . Thus, there are $5 - 3 = 2$ edges of each such color among the remaining eight vertices.

We now bound the number of partitions that do not produce two disjoint rainbow K_4 . For $\{c_1, \dots, c_6\}$, there are 3 matching edges among the remaining eight vertices. A 4-set on which color c_i appears twice is the vertex set of a pair of these edges, so each such color yields at most $\binom{3}{2} = 3$ non-rainbow 4-sets, for at most $6 \cdot 3 = 18$ in total. For $\{c_7, \dots, c_{11}\}$, there are exactly two matching edges among the remaining eight vertices, and these determine a unique non-rainbow K_4 . Thus, these five colors produce at most 5 additional K_4 that are not rainbow. Each K_4 that is not rainbow determines a unique unordered partition of the remaining eight vertices into two sets of size 4. Since the total number of bad partitions is at most $18 + 5 = 23 < 35$, we have at least one partition into two disjoint rainbow K_4 . $\square$

Lemma 3.22. *Let ρ be a minimum proper coloring of K_{11} . Suppose that $T \subseteq V(K_{11})$ is a valid triple and that $B_1, B_2 \subseteq V(K_{11})$ each induce a rainbow K_4 such that T, B_1, B_2 are pairwise disjoint. Then the union of the spanning trees of $K_{11}[T]$, $K_{11}[B_1]$, and $K_{11}[B_2]$ is a color-connecting set of size 8.*

Proof. Note that $P = \{T, B_1, B_2\}$ is a block partition of $V(K_{11})$, and by Proposition 3.14, it suffices to rule out the existence of seals. Recall that a seal requires an even number of vertices, so we need only consider the sets $\{B_1\}$, $\{B_2\}$, and $\{B_1, B_2\}$. By Lemma 3.15, a seal on $\{B_1\}$ or $\{B_2\}$ exists exactly when the corresponding K_4

is not rainbow. By Lemma 3.16, a seal on $\{B_1, B_2\} = P \setminus \{T\}$ exists exactly when T is invalid. So P is seal-free when T is valid and $K_{11}[B_1], K_{11}[B_2]$ are rainbow. Let W be the union of the spanning trees of $K_{11}[T], K_{11}[B_1]$, and $K_{11}[B_2]$, so $|W| = (3 - 1) + (4 - 1) + (4 - 1) = 8$. By Proposition 3.14, it follows that W is color-connecting when T is valid and $K_{11}[B_1], K_{11}[B_2]$ are rainbow. $\square$

Theorem 3.23. *Let $3 \leq n \leq 12$ and let ρ be a minimum proper coloring of K_n . Then*

$$\text{wld}_\rho(K_n) = \left\lfloor \frac{4\lceil n/2 \rceil}{3} \right\rfloor.$$

Proof. Given a minimum proper coloring ρ of K_n , Theorem 3.11 establishes that

$$\text{wld}_\rho(K_n) \geq \left\lfloor \frac{4\lceil n/2 \rceil}{3} \right\rfloor.$$

We show that this lower bound is sharp for $3 \leq n \leq 12$. We first consider odd n with $3 \leq n \leq 11$.

For $n \in \{3, 5\}$, the general bound $\text{wld}_\rho(K_n) \leq n - 1$ from Eq. (1) gives

$$\text{wld}_\rho(K_3) \leq 2 \text{ and } \text{wld}_\rho(K_5) \leq 4,$$

which equal the respective lower bounds.

For $n = 7$, by Lemma 3.12, there exists a valid triple $T \subseteq V(K_7)$. Let $B = V(K_7) \setminus T$, so $|B| = 4$. By Lemma 3.16, a seal of the form $(c_i, \{B\})$ exists if and only if T is invalid. Since T is valid, the partition $P = \{T, B\}$ is seal-free. Equivalently, by Lemma 3.15, $K_7[B]$ is rainbow. Choosing a spanning tree in each block forms a color-connecting set of size $(3 - 1) + (4 - 1) = 5$ by Lemma 3.18, so

$$\text{wld}_\rho(K_7) \leq 5.$$

For $n = 9$, by Lemma 3.19, there exists a partition of $V(K_9)$ into three disjoint valid triples T_1, T_2, T_3 . By Lemma 3.20, the union of the spanning trees of $K_9[T_1], K_9[T_2]$, and $K_9[T_3]$ forms a color-connecting set W of size $|W| = 3 \cdot 2 = 6$. Thus,

$$\text{wld}_\rho(K_9) \leq 6.$$

For $n = 11$, by Lemma 3.21, we have a partition of $V(K_{11})$ into T, B_1, B_2 where T is a valid triple and B_1, B_2 induce rainbow K_4 . By Lemma 3.22, T, B_1, B_2 form the components of a color-connecting set W for K_{11} with $|W| = 8$. Thus,

$$\text{wld}_\rho(K_{11}) \leq 8.$$

These inequalities agree with the respective lower bounds, completing the proof for odd n with $3 \leq n \leq 11$.

We now consider even n with $4 \leq n \leq 12$. Choose $v \in V(K_n)$, then, by Lemma 3.17, the restriction ρ_v is a minimum proper coloring of K_{n-1} and

$$\text{wld}_\rho(K_n) \leq \text{wld}_{\rho_v}(K_{n-1}) = \left\lfloor \frac{4\lceil (n-1)/2 \rceil}{3} \right\rfloor = \left\lfloor \frac{4\lceil n/2 \rceil}{3} \right\rfloor,$$

where the last equality holds since n is even. By the odd cases established above, the lower bound completes the proof for even n with $4 \leq n \leq 12$. $\square$

3.3.4 Weak Upper Bound

Lemma 3.24. *Let ρ be a minimum proper coloring of K_n with n odd and let $T \subseteq V(K_n)$ with $|T| = 3$.*

- (i) *If $v_i \notin T$, then K_n has an edge of color c_i with exactly one endpoint in T .*
- (ii) *If $T = \{v_i, a, b\}$, then K_n has an edge of color c_i with exactly one endpoint in T if and only if $\rho(ab) \neq c_i$.*

Proof. Since n is odd, M_i is a near-perfect matching covering every vertex except v_i . We seek an edge colored c_i with exactly one endpoint in T .

To prove (i), suppose $v_i \notin T$, so all three vertices of T are matched by M_i . Since a set of three vertices can contain at most one edge of the matching M_i , at least one vertex of T is matched by M_i to a vertex outside T .

To prove (ii), suppose $T = \{v_i, a, b\}$. The vertex v_i is isolated in M_i , while each of a and b is incident to exactly one edge of M_i . These edges have both of their endpoints in T if and only if a and b are adjacent to each other, that is, $ab \in M_i$, or $\rho(ab) = c_i$. If $\rho(ab) \neq c_i$, at least one of a, b shares an edge colored c_i with a vertex outside of T . If instead $\rho(ab) = c_i$, then ab is the only edge colored c_i incident to T , so there is no edge colored c_i with exactly one endpoint in T . $\square$

Theorem 3.25. *Let $n \geq 7$ be odd and let ρ be a minimum proper coloring of K_n . Then*

$$\text{wld}_\rho(K_n) \leq n - 2.$$

Proof. By Lemma 3.12, there exists a valid triple $T \subseteq V(K_n)$. Let $B = V(K_n) \setminus T$. Choose a spanning tree of $K_n[T]$ and a spanning tree of $K_n[B]$, and let W be the union of their edges. Then

$$|W| = (|T| - 1) + (|B| - 1) = 2 + (n - 4) = n - 2.$$

Fix a color c_i . If $v_i \notin T$, then Lemma 3.24(i) gives an edge of color c_i between the vertices of T and B . If $v_i \in T$, then the fact that T is a valid triple together with Lemma 3.24(ii) gives an edge of color c_i between the vertices of T and B .

The wild edges make T and B each connected, while the edge of color c_i guaranteed by Lemma 3.24 joins the two wild components. Therefore, G_i^W is connected. Since c_i was arbitrary, W is color-connecting and we have

$$\text{wld}_\rho(K_n) \leq n - 2. \quad \square$$

Corollary 3.26. *Let $n \geq 8$ be even, and let ρ be a minimum proper coloring of K_n . Then*

$$\text{wld}_\rho(K_n) \leq n - 3.$$

Proof. Let $v \in V(K_n)$. Combining Lemma 3.17 with Theorem 3.25, since $n - 1 \geq 7$ is odd, we have

$$\text{wld}_\rho(K_n) \leq \text{wld}_{\rho_v}(K_{n-1}) \leq (n - 1) - 2 = n - 3. \quad \square$$

3.3.5 Performance of the Greedy Algorithm

We conclude our study of minimum proper colorings of K_n considering Algorithm 5.10 of [2], an algorithm that produces a color-connecting set. Given a set W of edges already colored wild, the *dip number* of an edge e is

$$\delta_W(e) = \sum_{i=1}^{\ell} (\kappa(G_i^W) - \kappa(G_i^{W \cup \{e\}})),$$

the number of colors c_i for which coloring e wild merges two components of G_i^W . The greedy algorithm repeatedly selects an edge of maximum dip number until the selected set is color-connecting, breaking ties with the potential rule of [2]; our arguments do not depend on how ties are broken.

While the output of Algorithm 5.10 is not necessarily ideal, the color-connecting set produced is always a forest.

Lemma 3.27. *Let ρ be a minimum proper coloring of K_n , and let W_{greedy} be the output of Algorithm 5.10 of [2]. Then W_{greedy} is acyclic. Consequently, $|W_{\text{greedy}}| \leq n - 1$.*

Proof. Suppose the current selected set of edges is W and the algorithm has not yet terminated. Then there is a color c_i for which G_i^W is disconnected. Since K_n is complete, there is an edge uv joining two distinct components of G_i^W .

The edge uv cannot have color c_i , otherwise it would already belong to M_i , contradicting the assumption that its endpoints lie in distinct components of G_i^W . Thus, coloring uv wild merges two components of G_i^W , so uv has a positive dip number.

Therefore, whenever the algorithm has not terminated, an edge of positive dip number exists. Since the algorithm selects an edge of maximum dip number, it never selects an edge of dip number zero.

We claim that an edge of positive dip number joins two distinct components of the current selected wild set W . Suppose instead that the endpoints u and v of an edge were already connected by a path in W . Since $W \subseteq E(G_i^W)$ for every color c_i , the vertices u and v would already be in the same component of G_i^W for every color. Adding uv would therefore not merge monochromatic components, so its dip number would be zero.

Thus, every edge selected by the algorithm joins two distinct components of the current selected set. Adding such an edge cannot create a cycle. Since the algorithm begins with an empty set, W_{greedy} is acyclic.

As a forest on at most n vertices, it has at most $n - 1$ edges. Therefore, $|W_{\text{greedy}}| \leq n - 1$. $\square$

Lemma 3.28. *Let W be a color-connecting set for K_n . If $M_i \subseteq W$ for some color c_i , then the edges of W form a connected spanning subgraph of K_n . Consequently, $|W| \geq n - 1$.*

Proof. Since W is color-connecting, the spanning subgraph G_i^W is connected. By hypothesis, $M_i \subseteq W$, so $E(G_i^W) = M_i \cup W = W$. Every connected graph on n vertices has at least $n - 1$ edges, so $|W| \geq n - 1$. $\square$

Theorem 3.29. *Let ρ be a minimum proper coloring of K_n and let W_{greedy} be the output of Algorithm 5.10 of [2]. For $n \geq 3$, we have*

$$\frac{|W_{\text{greedy}}|}{\text{wld}_{\rho}(K_n)} \leq \frac{n - 1}{\left\lfloor \frac{4\lceil n/2 \rceil}{3} \right\rfloor},$$

and

$$0 \leq |W_{\text{greedy}}| - \text{wld}_{\rho}(K_n) \leq n - 1 - \left\lfloor \frac{4\lceil n/2 \rceil}{3} \right\rfloor.$$

In particular, this upper bound on the approximation ratio tends to $3/2$ as n tends to infinity.

Proof. By Lemma 3.27, $|W_{\text{greedy}}| \leq n - 1$. By Theorem 3.11, $\text{wld}_{\rho}(K_n) \geq \left\lfloor \frac{4\lceil n/2 \rceil}{3} \right\rfloor$. Combining these inequalities gives

$$\frac{|W_{\text{greedy}}|}{\text{wld}_{\rho}(K_n)} \leq \frac{n - 1}{\left\lfloor \frac{4\lceil n/2 \rceil}{3} \right\rfloor}.$$

Since W_{greedy} is color-connecting and $\text{wld}_{\rho}(K_n)$ is the minimum size of a color-connecting set, we have $|W_{\text{greedy}}| - \text{wld}_{\rho}(K_n) \geq 0$. The preceding upper and lower bounds also give

$$|W_{\text{greedy}}| - \text{wld}_{\rho}(K_n) \leq n - 1 - \left\lfloor \frac{4\lceil n/2 \rceil}{3} \right\rfloor.$$

Finally, since $\lceil 4\lceil n/2 \rceil/3 \rceil = 2n/3 + O(1)$, as $n \rightarrow \infty$, we have

$$\frac{n - 1}{\left\lfloor \frac{4\lceil n/2 \rceil}{3} \right\rfloor} \rightarrow \frac{3}{2}. \quad \square$$

The following corollary identifies color-class saturation as a structural failure of the greedy algorithm. When saturation occurs, the algorithm returns a spanning tree, which is the largest possible acyclic output, even though a smaller color-connecting set exists whenever $n \geq 7$.

Corollary 3.30. *Let ρ be a minimum proper coloring of K_n , and suppose that Algorithm 5.10 of [2] saturates a color class, that is, $M_i \subseteq W_{\text{greedy}}$ for some color c_i . Then*

$$|W_{\text{greedy}}| = n - 1.$$

In that case, W_{greedy} is not ideal whenever $n \geq 7$. More precisely,

$$|W_{\text{greedy}}| - \text{wld}_{\rho}(K_n) \geq \begin{cases} 1, & n \geq 7 \text{ odd}, \\ 2, & n \geq 8 \text{ even}. \end{cases}$$

Proof. By Lemma 3.28, $|W_{\text{greedy}}| \geq n - 1$. By Lemma 3.27, $|W_{\text{greedy}}| \leq n - 1$. Therefore, $|W_{\text{greedy}}| = n - 1$. Suppose that $n \geq 7$ is odd. By Theorem 3.25, $\text{wld}_\rho(K_n) \leq n - 2$. Thus,

$$|W_{\text{greedy}}| - \text{wld}_\rho(K_n) \geq (n - 1) - (n - 2) = 1.$$

Now suppose that $n \geq 8$ is even. By Corollary 3.26, $\text{wld}_\rho(K_n) \leq n - 3$. Therefore,

$$|W_{\text{greedy}}| - \text{wld}_\rho(K_n) \geq (n - 1) - (n - 3) = 2. \quad \square$$

4 Complete Bipartite Graphs

As in the previous section, we focus primarily on equitable 2-colorings and minimum proper colorings. Throughout, V_m and V_n denote the parts of $K_{m,n}$, with $|V_m| = m$ and $|V_n| = n$, and we write $K_{x,0}$ and $K_{0,x}$ for a subset of x isolated vertices of V_m and V_n , respectively. First, we make an observation about subgraphs of complete bipartite graphs.

Lemma 4.1. *Let G be a spanning subgraph of $K_{m,n}$, and define*

$$V_A = \{v \in V_m : d_G(v) \geq 1\} \quad \text{and} \quad V_B = \{v \in V_n : d_G(v) \geq 1\}.$$

Let $A = |V_A|$ and $B = |V_B|$. Then $|E(G)| \leq A(B - \kappa^*(G) + 1)$.

Proof. Let $(a_1, b_1), \dots, (a_{\kappa^*(G)}, b_{\kappa^*(G)})$ be the bipartition sizes of each nontrivial component of G where a_i is the number of vertices belonging to V_A and b_i is the number of vertices belonging to V_B . Since the nontrivial components partition $V_A \cup V_B$, we have

$$\sum_{i=1}^{\kappa^*(G)} a_i = A \quad \text{and} \quad \sum_{i=1}^{\kappa^*(G)} b_i = B.$$

Note that $|E(G)| \leq \sum_{i=1}^{\kappa^*(G)} a_i b_i$. Since every edge of G has an endpoint in V_n , each nontrivial component of G contains at least one vertex of V_B . For all $i \in [\kappa^*(G)]$, we then have that $b_i \leq B - (\kappa^*(G) - 1) = B - \kappa^*(G) + 1$ and so

$$|E(G)| \leq \sum_{i=1}^{\kappa^*(G)} a_i b_i \leq (B - \kappa^*(G) + 1) \sum_{i=1}^{\kappa^*(G)} a_i = A(B - \kappa^*(G) + 1). \quad \square$$

4.1 Equitable 2-Colorings

An equitable 2-coloring of $K_{m,n}$ has two color classes containing $\lfloor mn/2 \rfloor$ and $\lceil mn/2 \rceil$ edges. We obtain a sharp upper bound on $\text{wld}(K_{m,n})$ for such a coloring by considering the possible structures of disconnected color classes. We first construct a family of equitable 2-colorings ε_{2^*} (see Fig. 4.1).

Construction 4.2. Let $2 \leq m \leq n$. We construct an equitable 2-coloring ε_{2^*} such that every color class has $\lfloor n/2 \rfloor + 1$ connected components.

For even n , partition the vertices of V_n into two sets of size $n/2$. Color all edges incident to the first set with c_1 and all edges incident to the second set with c_2 . Each color class therefore induces a copy of $K_{m,n/2}$. In each color, the remaining $n/2$ vertices in V_n are isolated while the vertices incident to edges of each color form a single nontrivial component. Thus, each color class has $n/2 + 1$ connected components.

For odd n , partition the vertices of V_n into two sets of size $\lfloor n/2 \rfloor$ and one remaining vertex. Color all edges incident to the first set with c_1 and all edges incident to the second set with c_2 . For the remaining vertex in V_n , partition the edges equitably between c_1 and c_2 . Consequently, the color classes contain $\lfloor mn/2 \rfloor$ and $\lceil mn/2 \rceil$ edges, respectively. In each color, the remaining $\lfloor n/2 \rfloor$ vertices in V_n are isolated while the vertices incident to edges of each color form a single nontrivial component. Thus, each color class has $\lfloor n/2 \rfloor + 1$ connected components.

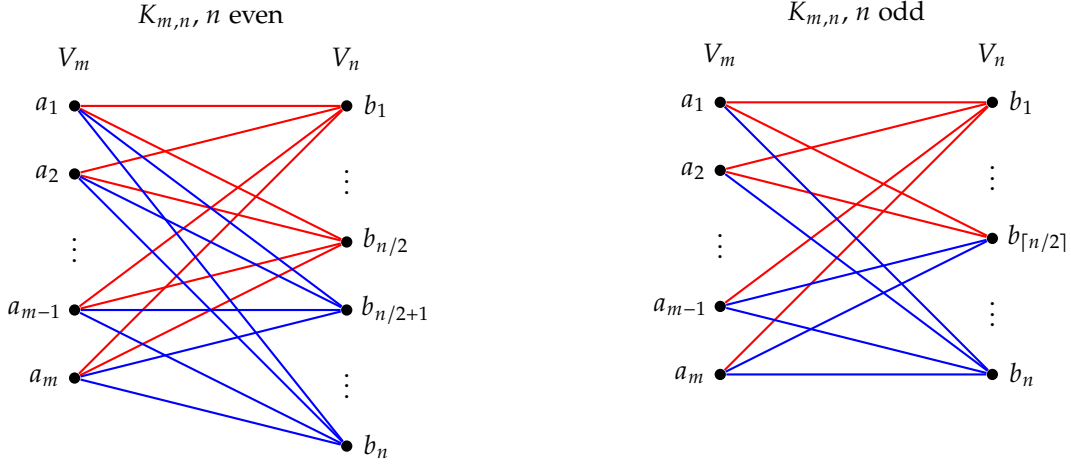


Figure 4.1: The equitable 2-colorings of $K_{m,n}$ attaining $\text{wld} = 2\lfloor n/2 \rfloor$.

From (2), for either parity of n , we have

$$\text{wld}_{\varepsilon_2^*}(K_{m,n}) = 2\left(\lfloor n/2 \rfloor + 1\right) - 2 = 2\lfloor n/2 \rfloor.$$

Theorem 4.3. For any equitable 2-coloring of the edges of $K_{m,n}$ with $2 \leq m \leq n$,

$$\text{wld}(K_{m,n}) \leq 2\lfloor n/2 \rfloor.$$

Moreover, this bound is sharp.

Proof. Let $G = K_{m,n}$ with an equitable 2-coloring. We will prove that $\sum_{i=1}^2 (\kappa(G_i) - 1) \leq 2\lfloor n/2 \rfloor$.

If G_1 and G_2 are connected, then $\sum_{i=1}^2 (\kappa(G_i) - 1) = 0 \leq 2\lfloor n/2 \rfloor$. We now consider cases where at least one of $\{G_1, G_2\}$ is disconnected. Let $(V_A)_i \subseteq V_m$ and $(V_B)_i \subseteq V_n$ be the vertices with degree at least one in color c_i , with $|(V_A)_i| = A_i$ and $|(V_B)_i| = B_i$.

Case 1. $A_1 < m$ and $B_1 < n$. Then G_1 has an isolated vertex in each part; let $v \in V_m$ and $w \in V_n$ be isolated in G_1 . In G_2 , v is adjacent to every vertex in V_n and w is adjacent to every vertex in V_m including v . Thus, $\kappa(G_2) = 1$. It remains to show $\kappa(G_1) - 1 \leq 2\lfloor n/2 \rfloor$.

We have $\kappa(G_1) = m - A_1 + n - B_1 + \kappa^*(G_1)$. Let $b = \lceil |E(G_1)|/A_1 \rceil$, the least integer x with $|E(G_1)| \leq A_1 x$. Note that $1 \leq b \leq n$. Consider the bipartite graph

$$G'_1 := K_{A_1, b} \cup K_{m-A_1, 0} \cup K_{0, n-b} \subseteq K_{m,n}.$$

Then, $\kappa(G'_1) = m - A_1 + n - b + 1$. We will show that $\kappa^*(G_1) \leq B_1 + 1 - b$, which implies $\kappa(G_1) \leq \kappa(G'_1)$.

By Lemma 4.1, $|E(G_1)| \leq A_1(B_1 - \kappa^*(G_1) + 1)$ and so $\frac{|E(G_1)|}{A_1} \leq B_1 - \kappa^*(G_1) + 1$. Since $B_1 - \kappa^*(G_1) + 1 \in \mathbb{N}$, $\lceil |E(G_1)|/A_1 \rceil \leq B_1 - \kappa^*(G_1) + 1$. By definition, $b = \lceil |E(G_1)|/A_1 \rceil$, which gives $b \leq B_1 - \kappa^*(G_1) + 1$, or equivalently $\kappa^*(G_1) \leq B_1 + 1 - b$ and thus $\kappa(G_1) \leq \kappa(G'_1)$.

Now, let $S := A_1 + b = n + m + 1 - \kappa(G'_1)$. Then $b = S - A_1$. We also have $|E(G_1)| \leq A_1 b$. Then

$$\begin{aligned} |E(G_1)| &\leq A_1 b = A_1(S - A_1) \leq \lfloor S^2/4 \rfloor = \left\lfloor \frac{(m + n + 1 - \kappa(G'_1))^2}{4} \right\rfloor \leq \frac{(m + n + 1 - \kappa(G'_1))^2}{4} \\ 4|E(G_1)| &\leq (m + n + 1 - \kappa(G'_1))^2 \\ \kappa(G'_1) &\leq m + n + 1 - 2\sqrt{|E(G_1)|} \quad \text{since } m + n + 1 - \kappa(G'_1) = S > 0. \end{aligned}$$

Since $|E(G_1)| \in \{\lfloor mn/2 \rfloor, \lceil mn/2 \rceil\}$, we then have

$$\kappa(G'_1) \leq m + n + 1 - 2\sqrt{\lfloor mn/2 \rfloor} \leq m + n + 1 - \sqrt{2(mn - 1)} \leq \begin{cases} m + n + 1 - m & \text{if } n \text{ is even} \\ m + n + 1 - (m + 1) & \text{if } n \text{ is odd} \end{cases} \leq 2\lfloor n/2 \rfloor + 1$$

where the second to last inequality follows from the fact that $\sqrt{2(mn - 1)} \geq m$ when n is even and $\sqrt{2(mn - 1)} \geq m + 1$ when n is odd for $2 \leq m \leq n$.

Thus, $\kappa(G_1) \leq \kappa(G'_1) \leq 2\lfloor n/2 \rfloor + 1$, so $\sum_{i=1}^2 (\kappa(G_i) - 1) \leq 2\lfloor n/2 \rfloor$. By symmetry, if either G_1 or G_2 satisfies the conditions of Case 1, we are done.

Case 2. $A_i = m$ or $B_i = n$ for each i . Suppose that $A_i = m$. Then

$$\kappa(G_i) = \kappa^*(G_i) + n - B_i. \quad (4)$$

By Lemma 4.1, $|E(G_i)| \leq m(B_i - \kappa^*(G_i) + 1)$. Since $|E(G_i)| \geq \lfloor mn/2 \rfloor$,

$$\begin{aligned} \lfloor mn/2 \rfloor &\leq m(B_i - \kappa^*(G_i) + 1) \\ \kappa^*(G_i) &\leq B_i + 1 - \frac{1}{m}\lfloor mn/2 \rfloor \end{aligned} \quad (5)$$

Combining (4) and (5), we have

$$\kappa(G_i) = \kappa^*(G_i) + n - B_i \leq B_i + 1 - \frac{1}{m}\lfloor mn/2 \rfloor + n - B_i = n + 1 - \frac{1}{m}\lfloor mn/2 \rfloor = \begin{cases} n + 1 - n/2 & \text{if } mn \text{ even} \\ n + 1 - n/2 + 1/(2m) & \text{if } mn \text{ odd} \end{cases}$$

For either parity of mn , since $m \geq 2$ and $\kappa(G_i)$ is an integer, we have

$$\kappa(G_i) \leq \lfloor n/2 \rfloor + 1$$

Now, suppose that $B_i = n$. A similar argument produces

$$\kappa(G_i) \leq \lfloor m/2 \rfloor + 1 \leq \lfloor n/2 \rfloor + 1$$

Thus, for $i = 1, 2$, we have $\kappa(G_i) \leq \lfloor n/2 \rfloor + 1$ so $\sum_{i=1}^2 (\kappa(G_i) - 1) \leq 2\lfloor n/2 \rfloor$, and we are done.

In all cases, we have $\sum_{i=1}^2 (\kappa(G_i) - 1) \leq 2\lfloor n/2 \rfloor$. By Proposition 2.1,

$$\text{wld}(G) \leq 2\lfloor n/2 \rfloor.$$

The sharpness of the bound is guaranteed by Construction 4.2. □

4.2 Minimum Proper Colorings

By König's edge-coloring theorem [6], the complete bipartite graph $K_{m,n}$ satisfies

$$\chi'(K_{m,n}) = \Delta(K_{m,n}).$$

For $m \leq n$, every minimum proper coloring of $K_{m,n}$ uses exactly n colors. Moreover, each color class is a matching of size m . For a color c_i , we write M_i for its monochromatic matching. We determine the wild number for minimum proper colorings of $K_{m,n}$ and show that it is independent of the choice of proper edge coloring.

Theorem 4.4. *Let $K_{m,n}$ be a complete bipartite graph with $2 \leq m \leq n$. For any minimum proper coloring ρ of $K_{m,n}$,*

$$\text{wld}_\rho(K_{m,n}) = n.$$

Proof. Let $M_1, \dots, M_n$ denote the color classes of $K_{m,n}$ with respect to ρ . Each M_i consists of m independent edges, forming m components of size 2. The remaining $n - m$ vertices are isolated, forming $n - m$ components of size 1. Thus, each color class has $m + (n - m) = n$ components. From Proposition 2.2, it follows that

$$\text{wld}_\rho(K_{m,n}) \geq \left\lceil \frac{\sum_{i=1}^n (\kappa(M_i) - 1)}{n - 1} \right\rceil = \frac{n(n - 1)}{n - 1} = n.$$

To prove the upper bound, take some $v \in V_m$ and let W consist of every edge incident to v . Note that $|W| = n$.

Consider any color class M_i . Since M_i is a matching of size m , M_i covers V_m . Thus, M_i contains exactly one edge incident to v . The remaining $n - 1$ edges of W connect v to every vertex in V_n that is not matched to v in M_i , including all $n - m$ vertices of V_n isolated in M_i . The vertex of V_n matched to v in M_i is already connected to v by its edge in M_i .

For any other vertex $v' \in V_m \setminus \{v\}$, M_i matches v' with some vertex $u' \in V_n$. Thus, the edge $u'v'$ is in M_i , while the edge $u'v$ belongs to W . Hence, u' and v' are connected to v in $M_i \cup W$. Therefore, every vertex of $K_{m,n}$ belongs to the same component of $M_i \cup W$. Since M_i was arbitrary, W is color-connecting. Together with the lower bound, W is ideal and

$$\text{wld}_\rho(K_{m,n}) = n. \quad \square$$

5 Cartesian Products

Throughout this section, we write ρ for an arbitrary minimum proper coloring of $G \square H$ using $\chi'(G \square H)$ colors. Recall that $|V(G \square H)| = |V(G)||V(H)|$ and $|E(G \square H)| = |E(G)||V(H)| + |V(G)||E(H)|$.

Definition 5.1. Fix minimum proper colorings ρ_1 and ρ_2 of G and H , respectively, with disjoint color sets. We define ρ^* as the edge coloring of $G \square H$ where $\rho^*((g, h)(g', h)) = \rho_1(gg')$ and $\rho^*((g, h)(g, h')) = \rho_2(hh')$.

This coloring is proper and uses $\chi'(G) + \chi'(H)$ colors, but, in general, is not minimum.

Lemma 5.2. For a minimum proper coloring of $G \square H$, there exists a color c_i such that $(G \square H)_i$ has at least

$$|V(G \square H)| - \frac{|E(G \square H)|}{\chi'(G \square H)}$$

connected components.

Proof. Let $c_1, \dots, c_{\chi'(G \square H)}$ be the colors of the minimum proper coloring and let m_i be the number of edges colored c_i . Each color class is a matching, so $\kappa((G \square H)_i) = m_i + (|V(G \square H)| - 2m_i) = |V(G \square H)| - m_i$ where

$$\sum_{i=1}^{\chi'(G \square H)} m_i = |E(G \square H)|.$$

By averaging, there exists some color c_i such that $m_i \leq |E(G \square H)| / \chi'(G \square H)$. Therefore,

$$\kappa((G \square H)_i) \geq |V(G \square H)| - \frac{|E(G \square H)|}{\chi'(G \square H)}. \quad \square$$

The bounds in this section are obtained by bounding from below the number of components of some color class. For an ℓ -edge-colored graph (G, γ) , we write

$$\text{clb}_\gamma(G) = \max_{1 \leq i \leq \ell} (\kappa(G_i) - 1).$$

Note that $\text{clb}_\gamma(G) \leq \text{wld}_\gamma(G)$ by Proposition 2.1.

5.1 Complete Graphs

Proposition 5.3. Let $n, m \geq 4$. Let ρ_1 and ρ_2 be any minimum proper colorings of K_n and K_m , respectively. Then, for $\rho' \in \{\rho, \rho^*\}$,

$$\text{wld}_{\rho_1}(K_n) + \text{wld}_{\rho_2}(K_m) < \text{wld}_{\rho'}(K_n \square K_m).$$

Proof. We consider the cases $\rho' = \rho$ and $\rho' = \rho^*$. By definition, $\text{wld}_{\rho_1}(K_n) \leq n - 1$ and $\text{wld}_{\rho_2}(K_m) \leq m - 1$.

Case 1. $\rho' = \rho$. We show that $\text{clb}_{\rho'}(K_n \square K_m) > \text{wld}_{\rho_1}(K_n) + \text{wld}_{\rho_2}(K_m)$. By Lemma 5.2, there exists a color whose corresponding subgraph has at least

$$|V(K_n \square K_m)| - \frac{|E(K_n \square K_m)|}{\chi'(K_n \square K_m)}.$$

components. Since $\Delta(K_n \square K_m) = n + m - 2$, we have $\chi'(K_n \square K_m) \geq n + m - 2$. Combining Lemma 5.2 with the definition of $\text{clb}_\gamma(G)$, we then have

$$\frac{nm}{2} - 1 = nm - \frac{\binom{n}{2}m + \binom{m}{2}n}{n + m - 2} - 1 \leq nm - \frac{\binom{n}{2}m + \binom{m}{2}n}{\chi'(K_n \square K_m)} - 1 \leq \text{clb}_{\rho'}(K_n \square K_m).$$

Observe that $n + m - 2 < nm/2 - 1$ since $n, m \geq 4$, thus

$$\text{wld}_{\rho_1}(K_n) + \text{wld}_{\rho_2}(K_m) \leq n + m - 2 < \frac{nm}{2} - 1 \leq \text{clb}_{\rho'}(K_n \square K_m) \leq \text{wld}_{\rho'}(K_n \square K_m).$$

Case 2. $\rho' = \rho^*$. Again, we show that $\text{clb}_{\rho'}(K_n \square K_m) > \text{wld}_{\rho_1}(K_n) + \text{wld}_{\rho_2}(K_m)$. Recall that when n is even each color class of K_n is a perfect matching of size $n/2$ and when n is odd each color class of K_n is a near-perfect matching of size $(n - 1)/2$. Thus, when n is even the subgraph corresponding to each color class of K_n contains $n/2$ components and when n is odd the subgraph corresponding to each color class of K_n contains $(n + 1)/2$ components. Under ρ^* , the subgraph associated with a color class from ρ_1 occurs in each of the m copies of K_n , so when n is even $\text{clb}_{\rho'}(K_n \square K_m) \geq nm/2 - 1$ and when n is odd $\text{clb}_{\rho'}(K_n \square K_m) \geq m(n + 1)/2 - 1$. Observe that $n + m - 2 < nm/2 - 1$ and $n + m - 2 < m(n + 1)/2 - 1$ since $n, m \geq 4$. For either parity of n , we then have

$$\text{wld}_{\rho_1}(K_n) + \text{wld}_{\rho_2}(K_m) < \text{clb}_{\rho'}(K_n \square K_m) \leq \text{wld}_{\rho'}(K_n \square K_m). \quad \square$$

5.2 Complete Bipartite Graphs

Proposition 5.4. Let $2 \leq m \leq n$ and $2 \leq s \leq t$. Let ρ_1 and ρ_2 be any minimum proper colorings of $K_{m,n}$ and $K_{s,t}$, respectively. Then, for $\rho' \in \{\rho, \rho^*\}$,

$$\text{wld}_{\rho_1}(K_{m,n}) \cdot \text{wld}_{\rho_2}(K_{s,t}) < \text{wld}_{\rho'}(K_{m,n} \square K_{s,t}).$$

Proof. We consider the cases $\rho' = \rho$ and $\rho' = \rho^*$. From Theorem 4.4, we have $\text{wld}_{\rho_1}(K_{m,n}) = n$ and $\text{wld}_{\rho_2}(K_{s,t}) = t$.

Case 1. $\rho' = \rho$. Note that the Cartesian product of bipartite graphs is again bipartite, so $\chi'(K_{m,n} \square K_{s,t}) = \Delta(K_{m,n} \square K_{s,t}) = \Delta(K_{m,n}) + \Delta(K_{s,t}) = n + t$ by König's edge-coloring theorem. Applying Lemma 5.2,

$$(s + t)(m + n) - \frac{mn(s + t) + st(m + n)}{n + t} - 1 \leq \text{clb}_{\rho'}(K_{m,n} \square K_{s,t}).$$

We show that $\text{wld}_{\rho_1}(K_{m,n}) \cdot \text{wld}_{\rho_2}(K_{s,t}) = nt < (s + t)(m + n) - \frac{mn(s + t) + st(m + n)}{n + t} - 1$.

Observe that

$$(s + t)(m + n) - \frac{mn(s + t) + st(m + n)}{n + t} - 1 - (nt) = \frac{sn^2 + mt^2 - n - t}{n + t},$$

so it is enough to show that $sn^2 + mt^2 - n - t = n(sn - 1) + t(mt - 1) > 0$ which is true for $m, n, s, t \geq 2$. It follows that

$$\text{wld}_{\rho_1}(K_{m,n}) \cdot \text{wld}_{\rho_2}(K_{s,t}) < \text{clb}_{\rho'}(K_{m,n} \square K_{s,t}) \leq \text{wld}_{\rho'}(K_{m,n} \square K_{s,t}).$$

Case 2. $\rho' = \rho^*$. Recall that a minimum proper coloring of $K_{m,n}$ uses n colors where each color class contains m edges, so the corresponding subgraph contains $m + (n - m) = n$ components for each color class. Under ρ^* , the subgraph associated with a color class from ρ_1 occurs in each of the $s + t$ copies of $K_{m,n}$, so $\text{clb}_{\rho'}(K_{m,n} \square K_{s,t}) \geq n(s + t) - 1$. Thus,

$$\text{wld}_{\rho_1}(K_{m,n}) \cdot \text{wld}_{\rho_2}(K_{s,t}) = nt < n(s + t) - 1 \leq \text{wld}_{\rho'}(K_{m,n} \square K_{s,t})$$

where the first inequality follows from the fact that $ns > 1$ for $n, s \geq 2$. □

6 Concluding Remarks

Theorem 3.23 shows that the wild number for K_n with a minimum proper coloring is exactly the lower bound for $3 \leq n \leq 12$. These results lead to the following conjecture.

Conjecture 6.1. *For every $n \geq 3$ and every minimum proper coloring ρ of K_n ,*

$$\text{wld}_{\rho}(K_n) = \left\lfloor \frac{4\lceil n/2 \rceil}{3} \right\rfloor.$$

Question 7.5 of [2] asks how far the greedy algorithm can deviate from the wild number. For a minimum proper coloring ρ of K_n , we showed in Lemma 3.27 that the greedy output is always a forest and therefore has at most $n - 1$ edges. Combining this with Theorem 3.11 gives

$$0 \leq |W_{\text{greedy}}| - \text{wld}_{\rho}(K_n) \leq n - 1 - \left\lfloor \frac{4\lceil n/2 \rceil}{3} \right\rfloor.$$

If the algorithm saturates a color class, then its output is a spanning tree and therefore has exactly $n - 1$ edges. By Corollary 3.30, this output is not ideal whenever $n \geq 7$, since smaller color-connecting sets exist by Theorem 3.25 and Corollary 3.26.

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